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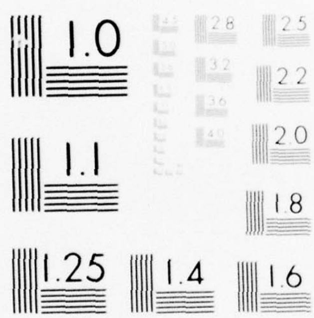
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ON THE SOLUTION OF A THREE-STAGE GAME REPRESENTING AN AGGREGATED AIR AND GROUND WAR

By: L. C. GOHEEN

Prepared for:

NAVAL ANALYSIS PROGRAMS (CODE 431)
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In order to develop a capability in BALFRAM to optimally allocate tactical air resources over N stages, three air-and-ground war models and the N-stage games given by them were studied. This report presents the results of that study.

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I INTRODUCTION

A. Background

During 1974, the Naval Warfare Research Center (NWRC) of Stanford Research Institute implemented the most advanced version of the Balanced Force Requirements Analysis Model (BALFRAM) at the Headquarters of the Commander in Chief Pacific. This advanced version incorporated several new capabilities, including an N-stage game for optimizing the allocation of tactical air resources and a capability for representing the effect on the military campaign of interdiction of logistic pipelines.

The N-stage game in BALFRAM was a state-of-the-art technique. Based on SABRE GRAND (ALPHA) [19], an algorithm developed by the Air Force to maximize the amount of ordnance delivered to the forward edge of the battle area, the N-stage game in BALFRAM was designed to provide significantly greater capability than the SABRE GRAND (ALPHA) algorithm by combining the algorithm with the BALFRAM simulation methodology to measure the effect of various allocations on the final outcome of the integrated land-air-sea campaign.

Basing BALFRAM's N-stage game on SABRE GRAND turned out to be an unfortunate choice. Operational use of the BALFRAM N-stage game at the Headquarters of the Commander in Chief Pacific revealed that its algorithm produced results that were, in some cases, demonstrably illogical, and the theoretical work of James Falk, Jerome Bracken, and others [1, 4, 5, 8, 11], revealed that the SABRE GRAND algorithm did not necessarily yield optimal solutions.

A proposal was therefore submitted to ONR on 24 June 1974 to review the structure of the N-stage game as it was then implemented in the BALFRAM computer program [15]. The proposal was accepted, and the first step of

the research approach was subsequently completed [9]. That step entailed review of the current structure of the N-stage game as it was implemented in the BALFRAM computer program, analysis of the underlying theoretical foundation of the N-stage game portion of the model, and comparisons of the intended structure of the model with actual computer program coding.

The conclusions deriving from the first step of the research were that the existing N-stage game formulation was incorrect, but that additional research could result in an effective formulation. Such additional research was to have constituted the second step of the research, and further effort was to lead to the integration of the corrected N-stage game into BALFRAM. However, because of the great complexity of the game's formulation and the paucity of documentation by the former subcontractor, only the first step of the research was completed.

Consequently, on 25 January 1975 a research task was proposed to ONR to continue research into the BALFRAM N-stage game in order to develop a valid and effective formulation that permits explicit measurement of the effects of interdiction of logistic pipelines on the integrated land-air-sea military campaign [14]. This proposal was also accepted and work was to begin on reformulating the N-stage game.

Consultant Dr. Melvin Dresher assisted in this research. Dr. Dresher had major criticisms of the theoretical foundation of the existing N-stage game algorithm, which, in his opinion, completely invalidated it. Research was therefore reoriented, as discussed in an SRI/NWRC letter to ONR [18].

Research was directed to existing methods for solving N-stage air war games so that one might be incorporated in BALFRAM to provide BALFRAM with the capability of optimizing the allocation of air resources over N stages [10].

In all, six methods were studied: two iterative methods, Lagrange dynamic programming [17], and the method of Ostermann and Boudreau [16]; one method based on linear programming, OPTSA I [6]; and three methods based on dynamic programming, OPTSA II [6], DYGAM [12], and that of Berkovitz and Dresher [2,3]. Our findings showed that two-sided Lagrange dynamic programming and the method of Ostermann and Boudreau could yield incorrect solutions to N-stage games. Also, incorporating OPTSA I or OPTSA II in BALFRAM was computationally infeasible, and there was no guarantee that the strategies produced by DYGAM were optimal or even nearly optimal. These conclusions are discussed in detail in Reference [10]. See also Reference [7].

In contrast, the method used by Berkovitz and Dresher appeared the most apt to solve the N-stage game in BALFRAM, and we have proceeded with that method. In brief, this method could be called two-sided dynamic programming over a continuous strategy space. It can be applied to N-stage games possessing a continuous and additive ($1 \le n \le N$) payoff function, continuous transition functions, and a continuum of strategies. Berkovitz and Dresher were able to show that such N-stage games could be solved by the solution of a sequence of N one move games, although the solution of these one move games could be quite complicated. See Reference [3]. It is important to realize that the method used by Berkovitz and Dresher requires the closed form solution of each of the N one move games. In the opinion of Dr. Dresher, the method was not amenable to numerical (i.e., computer) procedure. The reasoning was as follows: the game being solved had a continuum of strategies, i.e., was a continuous game. We sought optimal strategies for the N-stage game so that they might be implemented in BALFRAM. Numerical solution would necessitate considering only a finite number of strategies, i.e., solving a finite game. Thus the numerical procedure could solve the game only if the pure strategies entering into the optimal mixed

strategies of the continuous game were included among the strategies considered by the numerical procedure. See Reference [7] for a discussion of this point and an example.

B. Summary

It is clear that a formulation of an N-stage game incorporating the full detail of the BALFRAM simulation is computationally intractable. Thus a computationally tractable (and hence more aggregated) air-and-ground-war model must be developed to represent the much more complex BALFRAM system, air allocations in the more aggregated model must be optimized via an N-stage game, finally a method must be developed by which the strategies thus determined can be transformed into strategies implementable in BALFRAM. Finally, it is necessary to verify that the aggregated model accurately represents the BALFRAM system; or, alternatively, the strategies thus implemented must be tested for optimality in BALFRAM.

Therefore, the first task was the development of a computationally tractable air-and-ground-war model that would include explicit representation of ground combat; logistic pipelines; and the air missions of close air support, airfield neutralization, interdiction, air superiority, and air defense. This task did not result in the development of only one air-and-ground-war model but rather of a sequence of such models.

The first air-and-ground-war model (Model I) satisfied all our conditions save one, computational tractability, since the optimization of air allocations in this model required optimization over 14 variables, 7 for B and 7 for R. Since the solution method required that the game be solved in closed form, solving the N-stage game given by Model I was too formidable a task.

Consequently, Model II was formulated, which differed from Model I in that the battlefield was aggregated into three regions: T_B, the region defended by B and attacked by R; T_R, the region defended by R and attacked by B; and T_{BR}, the region that may be both defended and attacked by B and R. Under this aggregation scheme, B and R allocate airplanes to the three regions at N decision points. Airplanes allocated by B(R) to region $T_B(T_R)$ are suballocated between B(R)'s air-defense-versus-airfield-neutralization and air-defense-versus-interdiction missions. Airplanes allocated by B(R) to region $T_R(T_B)$ are suballocated between B(R)'s airfield-neutralization-and-interdiction missions. Airplanes allocated by B(R) to region T_{BR} are suballocated between B(R)'s air-strike, close-air-support, and air-defense-versus-close-air-support missions. The suballocations are computed independently in each of the regions T_B, T_R, T_{BR}, and at each decision point.

Model II and Model I both explicitly represent ground combat and logistic pipelines. As the solution of the N-stage game given by Model II progressed, it became apparent that this explicit representation greatly complicated the payoff function and calculations. In fact, the complete solution of the game appeared impracticable.

To simplify the situation, Model III (the surrogate model) was developed, in which the explicit representation of the ground war was replaced by surrogate factors representing the impact of the air support missions on the ground war. The remainder of the research effort was then spent on the solution of the N-stage game given by Model III, which has yet to be completed. Needless to say, important features of the earlier models have been sacrificed. Once Model III is solved, we must transform the optimal strategies thus derived into strategies implementable in BALFRAM. The method of accomplishing this has yet to be completely specified.

II MODEL I

Model I represents an aggregated air and ground war of fixed duration T, see Figure 1. Stage n, $1 \le n \le N$ begins at a fixed time t^{n-1} and ends at a fixed time t^n. Attrition processes are modeled by the Lanchester square law. (The square law was chosen to model air-to-air attrition, despite a possible lack of realism, because it seemed more computationally tractable than the linear law.) The air missions considered are those of airfield neutralization (AN), close air support (CAS), interdiction (I), air superiority (AS), air defense against airfield neutralization (AD vs AN), air defense against close air support (AD vs CAS), and air defense against interdiction (AD vs I). At each time t^n, $0 \le n < N$, the air-to-ground fire of B(R)'s AN mission is assumed to be uniformly allocated among R(B)'s seven missions.

An air allocation for B(R) in stage n is a vector $U_n(V_n) \in \mathbb{R}^7$, where $U_n = (U_{ni})$, $V_n = (V_{ni})$, $U_{ni}, V_{ni} \ge 0$ and $\sum_i U_{ni} = \sum V_{ni} = 1$. U_{n1}, U_{n2}, U_{n3}, and U_{n7} (V_{n1}, V_{n2}, V_{n3} and V_{n7}) are the fractions of available airplanes at time t^{n-1} assigned by B(R) to the missions of AN, CAS, I and AS, respectively. U_{n4}, U_{n5}, and U_{n6} (V_{n4}, V_{n5}, and V_{n6}) are the fractions of available airplanes at time t^{n-1} assigned by B(R) to the missions of AD vs AN, AD vs CAS, and AD vs I, respectively. Attack airplanes (i.e., airplanes assigned to the AN, CAS, and I missions) are assumed to perform their missions with equal effectiveness whether or not under attack by AD airplanes. Interdiction of B(R)'s logistic pipeline in stage n degrades the effectiveness of B(R) ground forces in stage $n+1$. The effect of the CAS missions on the ground war is modeled by exogenous firepower in the differential equations modeling the ground war.

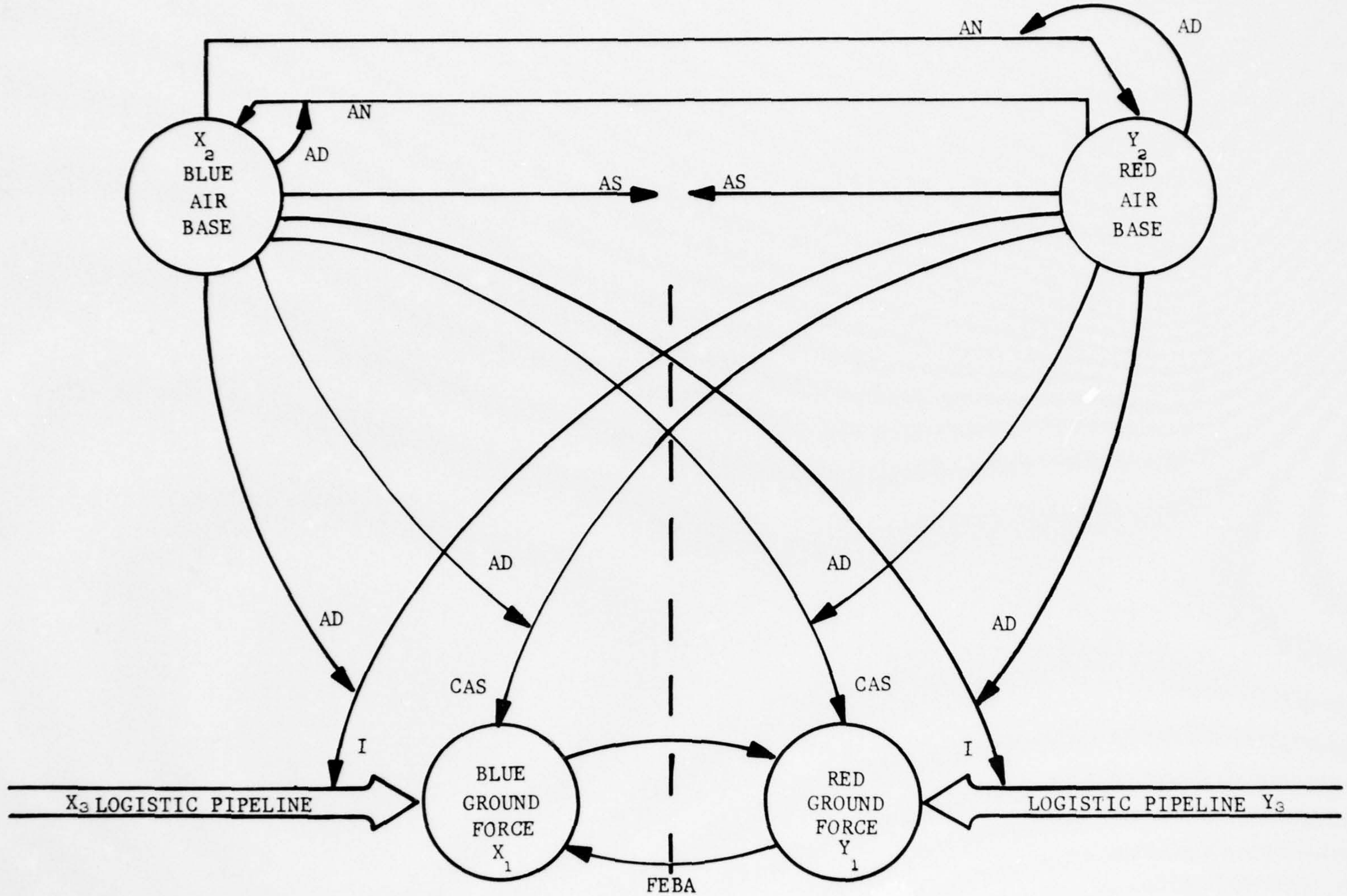

FIGURE 1. GRAPHIC REPRESENTATION OF MODEL I.

We let $X_1(t)$, $X_2(t)$, $X_3(t)$ $\left(Y_1(t), Y_2(t), Y_3(t)\right)$ be the order of battle of B(R)'s ground force, air force, and pipeline, respectively, at time t. Let $x^i(t)$ $\left(y^i(t)\right)$, $1 \le i \le 3$, be the order of battle of B(R)'s AN, CAS, and I mission, respectively, at time t; let $x^i(t)\left(y^i(t)\right)$, $4 \le i \le 6$ be the order of battle of B(R)'s AD vs AN, AD vs CAS, and AD vs I mission, respectively, at time t; let $x^7(t)\left(y^7(t)\right)$ be the order of battle of B(R)'s AS mission at time t. Let $\xi_n^B\left(\xi_n^R\right)$ be the regeneration rate for B(R)'s pipeline during stage n.

The FEBA movement rate, φ, at time $t\epsilon\left[t^{n-1}, t^n\right]$ is assumed to be an integrable function of $X_i(t)$ and $Y_i(t)$, $1 \le i \le 3$, U_n and V_n. The distance of advance in stage n is then

$$\varphi_n = \int_{t^{n-1}}^{t^n} \varphi(t)dt \quad .$$

A. Differential Equations

The following system of differential equations, representing Model I, then arises for stage n, $1 \le n \le N$. Throughout $t\epsilon\left[t^{n-1}, t^n\right]$

Ground War

$$\varphi_n = \int_{t^{n-1}}^{t^n} \varphi(t)dt$$

$$\begin{bmatrix} \dot{X}_1(t) \\ \dot{Y}_1(t) \end{bmatrix} = \begin{bmatrix} 0 & -A_1(n,U_{n-1},V_{n-1}) \\ -B_1(n,U_{n-1},V_{n-1}) & 0 \end{bmatrix} \begin{bmatrix} X_1(t) \\ Y_1(t) \end{bmatrix} - \begin{bmatrix} a_1(n,V_n) \\ b_1(n,U_n) \end{bmatrix}$$

$$X_1(t),\ Y_1(t) \ge 0,\ X_1(0) \equiv X_1,\ Y_1(0) \equiv Y_1 \quad .$$

Logistic Pipelines

$$\dot{X}_3(t) = \xi_n^B - A^{**}y^3(t) \quad ,$$

$$\dot{Y}_3(t) = \xi_n^R - B^{**}x^3(t) \quad ,$$

$$X_3(t),\ Y_3(t) \geq 0,\ X_3(0) \equiv X_3,\ Y_3(0) \equiv Y_3 \quad .$$

Air War

Airfield Neutralization

$$\begin{bmatrix} \dot{x}^1(t) \\ \dot{x}^4(t) \\ \dot{y}^1(t) \\ \dot{y}^4(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & -A^*/7 & -A^4 \\ 0 & 0 & -(A^1 + A^*/7) & 0 \\ -B^*/7 & -B^4 & 0 & 0 \\ -(B^1 + B^*/7) & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x^1(t) \\ x^4(t) \\ y^1(t) \\ y^4(t) \end{bmatrix} ,$$

$$x^1(t^{n-1}) = U_{n1}X_2(t^{n-1}),\ x^4(t^{n-1}) = U_{n4}X_2(t^{n-1}),$$

$$y^1(t^{n-1}) = V_{n1}Y_2(t^{n-1}),\ y^4(t^{n-1}) = V_{n4}Y_2(t^{n-1}),$$

$$x^1(t),\ x^4(t),\ y^1(t),\ y^4(t) \geq 0 \quad .$$

Close Air Support

$$\begin{bmatrix} \dot{x}^2(t) \\ \dot{y}^5(t) \end{bmatrix} = \begin{bmatrix} 0 & -A^5 \\ -B^2 & 0 \end{bmatrix} \begin{bmatrix} x^2(t) \\ y^5(t) \end{bmatrix} - \frac{1}{7} \begin{bmatrix} A^*y^1(t) \\ B^*x^1(t) \end{bmatrix} ,$$

$$x^2(t^{n-1}) = U_{n2}X_2(t^{n-1}), \; y^5(t^{n-1}) = V_{n5}Y_2(t^{n-1}),$$

$$x^2(t), \; y^5(t) \geq 0 \quad .$$

$$\begin{bmatrix} \dot{y}^2(t) \\ \dot{x}^5(t) \end{bmatrix} = \begin{bmatrix} 0 & -B^5 \\ -A^2 & 0 \end{bmatrix} \begin{bmatrix} y^2(t) \\ x^5(t) \end{bmatrix} - \frac{1}{7} \begin{bmatrix} B^*x^1(t) \\ A^*y^1(t) \end{bmatrix} ,$$

$$y^2(t^{n-1}) = V_{n2}Y_2(t^{n-1}), \; x^5(t^{n-1}) = U_{n2}X_2(t^{n-1}),$$

$$y^2(t), \; x^5(t) \geq 0 \quad .$$

Interdiction

$$\begin{bmatrix} \dot{x}^3(t) \\ \dot{y}^6(t) \end{bmatrix} = \begin{bmatrix} 0 & -A^6 \\ -B^3 & 0 \end{bmatrix} \begin{bmatrix} x^3(t) \\ y^6(t) \end{bmatrix} - \frac{1}{7} \begin{bmatrix} A^*y^1(t) \\ B^*x^1(t) \end{bmatrix} ,$$

$$x^3(t^{n-1}) = U_{n3}X_2(t^{n-1}), \; y^6(t^{n-1}) = V_{n3}Y_2(t^{n-1}),$$

$$x^3(t), \; y^6(t) \geq 0 \quad .$$

$$\begin{bmatrix} \dot{y}^3(t) \\ \dot{x}^6(t) \end{bmatrix} = \begin{bmatrix} 0 & -B^6 \\ -A^3 & 0 \end{bmatrix} \begin{bmatrix} y^3(t) \\ x^6(t) \end{bmatrix} - \frac{1}{7} \begin{bmatrix} B^*x^1(t) \\ A^*y^1(t) \end{bmatrix} ,$$

$$y^3(t^{n-1}) = V_{n3}Y_2(t^{n-1}), \; x^6(t^{n-1}) = U_{n6}X_2(t^{n-1}),$$

$$y^3(t), \; x^6(t) \geq 0 \quad .$$

Air Superiority

$$\begin{bmatrix} \dot{x}^7(t) \\ \dot{y}^7(t) \end{bmatrix} = \begin{bmatrix} 0 & -A^7 \\ -B^7 & 0 \end{bmatrix} \begin{bmatrix} x^7(t) \\ y^7(t) \end{bmatrix} - \frac{1}{7} \begin{bmatrix} A^* y^1(t) \\ B^* x^1(t) \end{bmatrix} ,$$

$$x^7(t^{n-1}) = U_{n7}X_2(t^{n-1}), \; y^7(t^{n-1}) = V_{n7}Y_2(t^{n-1}),$$

$$x^7(t), \; y^7(t) \geq 0 \quad .$$

$$X_2(0) \equiv X_2, \; Y_2(0) \equiv Y \quad .$$

Modeling $A_1(n,U_{n-1},V_{n-1})$ and $B_1(n,U_{n-1},V_{n-1})$

We assume

$$A_1(n,U_{n-1},V_{n-1}) = f_n^R(U_{n-1},V_{n-1},\rho_R(t^{n-1}))A_1'(n)$$

and

$$B_1(n,U_{n-1},V_{n-1}) = f_n^B(U_{n-1},V_{n-1},\rho_B(t^{n-1}))B_1'(n),$$

where

$A_1'(n)\left(B_1'(n)\right)$ = the (index of combat effectiveness†) × (base attrition factor†) for B(R)'s ground force in stage n .

$\rho_B(t^{n-1})\left(\rho_R(t^{n-1})\right)$ = the resupply factor for B(R)'s ground force at time t^{n-1} ,

†See Glossary of [13] for definitions.

and

$$f_n^B(U_{n-1},V_{n-1},\rho_B(t^{n-1}))\left(f_n^R(U_{n-1},V_{n-1},\rho_R(t^{n-1}))\right) = \text{a non-negative}$$

constant modeling the effect of interdiction on B(R)'s ground force in stage n .

It remains to specify f_n^B and f_n^R.

Let x(y) = the throughput capacity of each component of B(R)'s pipeline unit. Thus $xX_3(t)\left(yY_3(t)\right)$ = the pipeline capacity for B(R) at time t . We take

$$f_n^B(U_{n-1},V_{n-1},\rho_B(t^{n-1})) = \frac{\rho_B(t^{n-1}) \wedge xX_3(t^{n-1})}{\rho_B(t^{n-1})}$$

and

$$f_n^R(U_{n-1},V_{n-1},\rho_R(t^{n-1})) = \frac{\rho_R(t^{n-1}) \wedge yY_3(t^{n-1})}{\rho_R(t^{n-1})} .$$

Note:

$$a \wedge b = \min(a,b) \quad , \quad a \vee b = \max(a,b) \quad ;$$

$$a \wedge bc = a \wedge (bc) \quad , \quad a \vee bc = a \vee (bc) \quad ;$$

$$a \wedge b/c = a \wedge (b/c) \quad , \quad a \vee b/c = a \vee (b/c) \quad ;$$

$$a \wedge b \pm c = (a \wedge b) \pm c \quad , \quad a \vee b \pm c = (a \vee b) \pm c \quad .$$

Modeling $a_1(n,V_n)$ and $b_1(n,U_n)$

One possibility is

$$a_1(n,V_n) = a_1'(n)V_{n2}Y_2(t^{n-1})$$

$$b_1(n,U_n) = b_1'(n)U_{n2}X_2(t^{n-1}) \quad .$$

B. The Formulated Game

B(R) picks $U = (U_n)\left(V = (V_n)\right)$ to max(min) the payoff function

$$\left[X_1(T) - Y_1(T)\right]$$

so that for $1 \le n \le N$, $t^{n-1} \le t \le t^n$

$$\begin{bmatrix} \dot{X}_1(t) \\ \\ \dot{Y}_1(t) \end{bmatrix} = \begin{bmatrix} 0 & -\dfrac{\left(\rho_R(t^{n-1}) \wedge yY_3(t^{n-1})\right)}{\rho_R(t^{n-1})} A'(n) \\ \\ -\dfrac{\left(\rho_B(t^{n-1}) \wedge xX_3(t^{n-1})\right)}{\rho_B(t^{n-1})} B'(n) & 0 \end{bmatrix} \begin{bmatrix} X_1(t) \\ \\ Y_1(t) \end{bmatrix}$$

$$- \begin{bmatrix} a_1'(n) V_{n2} Y_2(t^{n-1}) \\ \\ b_1'(n) U_{n2} X_2(t^{n-1}) \end{bmatrix} \quad ;$$

$$\dot{X}_3(t) = \xi_n^B - A^{**}y^3(t),$$

$$\dot{Y}_3(t) = \xi_n^R - B^{**}x^3(t);$$

$$\begin{bmatrix} \dot{x}^1(t) \\ \dot{x}^4(t) \\ \dot{y}^1(t) \\ \dot{y}^4(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & -A^*/7 & -A^* \\ 0 & 0 & -(A^1 + A^*/7) & 0 \\ -B^*/7 & -B^4 & 0 & 0 \\ -(B^1 + B^*/7) & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x^1(t) \\ x^4(t) \\ y^1(t) \\ y^4(t) \end{bmatrix} \quad ;$$

$$\begin{bmatrix} \dot{x}^2(t) \\ \dot{y}^5(t) \end{bmatrix} = \begin{bmatrix} 0 & -A^5 \\ -B^2 & 0 \end{bmatrix} \begin{bmatrix} x^2(t) \\ y^5(t) \end{bmatrix} - \frac{1}{7} \begin{bmatrix} A^*y^1(t) \\ B^*x^1(t) \end{bmatrix} \quad ;$$

$$\begin{bmatrix} \dot{y}^2(t) \\ \dot{x}^5(t) \end{bmatrix} = \begin{bmatrix} 0 & -B^5 \\ -A^2 & 0 \end{bmatrix} \begin{bmatrix} y^2(t) \\ x^5(t) \end{bmatrix} - \frac{1}{7} \begin{bmatrix} B^*x^1(t) \\ A^*y^1(t) \end{bmatrix} \quad ;$$

$$\begin{bmatrix} \dot{x}^3(t) \\ \dot{y}^6(t) \end{bmatrix} = \begin{bmatrix} 0 & -A^6 \\ -B^3 & 0 \end{bmatrix} \begin{bmatrix} x^3(t) \\ y^6(t) \end{bmatrix} - \frac{1}{7} \begin{bmatrix} A^*y^1(t) \\ B^*x^1(t) \end{bmatrix} \quad ;$$

$$\begin{bmatrix} \dot{y}^3(t) \\ \dot{x}^6(t) \end{bmatrix} = \begin{bmatrix} 0 & -B^6 \\ -A^3 & 0 \end{bmatrix} \begin{bmatrix} y^3(t) \\ x^6(t) \end{bmatrix} - \frac{1}{7} \begin{bmatrix} B^*x^1(t) \\ A^*y^1(t) \end{bmatrix} \quad ;$$

$$\begin{bmatrix} \dot{x}^7(t) \\ \dot{y}^7(t) \end{bmatrix} = \begin{bmatrix} 0 & -A^7 \\ -B^7 & 0 \end{bmatrix} \begin{bmatrix} x^7(t) \\ y^7(t) \end{bmatrix} - \frac{1}{7} \begin{bmatrix} A^*y^1(t) \\ B^*x^1(t) \end{bmatrix} \quad ;$$

$X_1(t),\ Y_1(t),\ X_3(t),\ Y_3(t),\ x^i(t),\ y^i(t) \geq 0;$

$x^i(t^{n-1}) = U_{ni}X_2(t^{n-1}),\ y^i(t^{n-1}) = V_{ni}Y_2(t^{n-1});$ and

$x_i(0) \equiv X_i,\ Y_i(0) \equiv Y_i \quad .$

C. The Solution of the Differential Equations

Throughout $t \geq t^{n-1}$. It is well known that systems of differential equations of the form

$$\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} 0 & -a \\ -b & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} a_1(t) \\ b_1(t) \end{bmatrix} \quad ,$$

$x(t^{n-1}),\ y(t^{n-1})$ given

have the solution

$$\begin{bmatrix} x(t) \\ \\ y(t) \end{bmatrix} = \begin{bmatrix} \cosh\left(\sqrt{ab}\ (t-t^{n-1})\right) & -\sqrt{\frac{a}{b}}\sinh\left(\sqrt{ab}\ (t-t^{n-1})\right) \\ \\ -\sqrt{\frac{b}{a}}\sinh\left(\sqrt{ab}\ (t-t^{n-1})\right) & \cosh\left(\sqrt{ab}\ (t-t^{n-1})\right) \end{bmatrix} \begin{bmatrix} x(t^{n-1}) \\ \\ y(t^{n-1}) \end{bmatrix}$$

$$+ \int_{t^{n-1}}^{t} \begin{bmatrix} \cosh\left(\sqrt{ab}\ (t-\sigma)\right) & -\sqrt{\frac{a}{b}}\sinh\left(\sqrt{ab}\ (t-\sigma)\right) \\ \\ -\sqrt{\frac{b}{a}}\sinh\left(\sqrt{ab}\ (t-\sigma)\right) & \cosh\left(\sqrt{ab}\ (t-\sigma)\right) \end{bmatrix} \begin{bmatrix} a_1(\sigma) \\ \\ b_1(\sigma) \end{bmatrix} d\sigma \quad .$$

Specifically,

$$\begin{bmatrix} \dot{X}_1(t) \\ \\ \dot{Y}_1(t) \end{bmatrix} = \begin{bmatrix} 0 & -\dfrac{\left(\rho_R(t^{n-1}) \wedge yY_3(t^{n-1})\right)A'(n)}{\rho_R(t^{n-1})} \\ \\ -\dfrac{\left(\rho_B(t^{n-1}) \wedge xX_3(t^{n-1})\right)B'(n)}{\rho_B(t^{n-1})} & 0 \end{bmatrix} \begin{bmatrix} X_1(t) \\ \\ Y_1(t) \end{bmatrix}$$

$$- \begin{bmatrix} a_1'(n)V_{n2}Y_2(t^{n-1}) \\ b_1'(n)U_{n2}X_2(t^{n-1}) \end{bmatrix}$$

$X_1(t^{n-1}),\ Y_1(t^{n-1})$

has the solution

$$X_1(t) = \left(X_1(t^{n-1}) + \frac{b_1'(n)U_{n2}X_2(t^{n-1})\rho_B(t^{n-1})}{\left(\rho_B(t^{n-1}) \wedge xX_3(t^{n-1})\right)B'(n)}\right)$$

$$\times \cosh\left(\left[\frac{\left(\rho_R(t^{n-1}) \wedge yY_3(t^{n-1})\right)A'(n)\left(\rho_B(t^{n-1}) \wedge xX_3(t^{n-1})\right)B'(n)}{\rho_R(t^{n-1})\rho_B(t^{n-1})}\right]^{\frac{1}{2}}(t-t^{n-1})\right)$$

$$- \left(Y_1(t^{n-1}) \left[\frac{\left(\rho_R(t^{n-1}) \wedge yY_3(t^{n-1})\right) A'(n) \rho_B(t^{n-1})}{\rho_R(t^{n-1}) \left(\rho_B(t^{n-1}) \wedge xX_3(t^{n-1})\right) B'(n)} \right]^{\frac{1}{2}} \right.$$

$$\left. + \frac{a_1'(n) V_{n2} Y_2(t^{n-1})}{\left[\frac{\left(\rho_R(t^{n-1}) \wedge yY_3(t^{n-1})\right) A'(n) \left(\rho_B(t^{n-1}) \wedge xX_3(t^{n-1})\right) B'(n)}{\rho_R(t^{n-1}) \rho_B(t^{n-1})} \right]^{\frac{1}{2}}} \right)$$

$$\times \sinh \left(\left[\frac{\left(\rho_R(t^{n-1}) \wedge yY_3(t^{n-1})\right) A'(n) \left(\rho_B(t^{n-1}) \wedge xX_3(t^{n-1})\right) B'(n)}{\rho_R(t^{n-1}) \rho_B(t^{n-1})} \right]^{\frac{1}{2}} (t - t^{n-1}) \right)$$

$$- \frac{b_1'(n) U_{n2} X_2(t^{n-1}) \rho_B(t^{n-1})}{\left(\rho_B(t^{n-1}) \wedge xX_3(t^{n-1})\right) B'(n)} \quad ,$$

$$Y_1(t) = \left(Y_1(t^{n-1}) + \frac{a_1'(n) V_{n2} Y_2(t^{n-1}) \rho_R(t^{n-1})}{\left(\rho_R(t^{n-1}) \wedge yY_3(t^{n-1})\right) A'(n)} \right)$$

$$\times \cosh \left(\left[\frac{\left(\rho_R(t^{n-1}) \wedge yY_3(t^{n-1})\right) A'(n) \left(\rho_B(t^{n-1}) \wedge xX_3(t^{n-1})\right) B'(n)}{\rho_R(t^{n-1}) \rho_B(t^{n-1})} \right]^{\frac{1}{2}} (t - t^{n-1}) \right)$$

$$- \left(X_1(t^{n-1}) \left[\frac{\left(\rho_B(t^{n-1}) \wedge xX_3(t^{n-1})\right) B'(n) \rho_R(t^{n-1})}{\rho_B(t^{n-1}) \left(\rho_R(t^{n-1}) \wedge yY_3(t^{n-1})\right) A'(n)} \right]^{\frac{1}{2}} \right.$$

$$\left. + \frac{b_1'(n) U_{n2} X_2(t^{n-1})}{\left[\frac{\left(\rho_R(t^{n-1}) \wedge yY_3(t^{n-1})\right) A'(n) \left(\rho_B(t^{n-1}) \wedge xX_3(t^{n-1})\right) B'(n)}{\rho_R(t^{n-1}) \rho_B(t^{n-1})} \right]^{\frac{1}{2}}} \right)$$

$$\times \sinh\left(\left[\frac{\left(\rho_R(t^{n-1}) \wedge yY_3(t^{n-1})\right)A'(n)\left(\rho_B(t^{n-1}) \wedge xX_3(t^{n-1})\right)B'(n)}{\rho_R(t^{n-1})\rho_B(t^{n-1})}\right]^{\frac{1}{2}}(t-t^{n-1})\right)$$

$$-\frac{a_1'(n)V_{n2}Y_2(t^{n-1})\rho_R(t^{n-1})}{\left(\rho_R(t^{n-1}) \wedge yY_3(t^{n-1})\right)A'(n)} \quad .$$

The differential equation

$$\dot{X}_3(t) = \xi_n^B - A^{**}y^3(t) \quad , \quad X_3(t^{n-1})$$

has the solution

$$X_3(t) = X_3(t^{n-1}) + \xi_n^B(t-t^{n-1}) - A^{**}\int_{t^{n-1}}^{t} y^3(s)ds \quad ,$$

while

$$\dot{Y}_3(t) = \xi_n^R - B^{**}x^3(t) \quad , \quad Y_3(t^{n-1})$$

has the solution

$$Y_3(t) = Y_3(t^{n-1}) + \xi_n^R(t-t^{n-1}) - B^{**}\int_{t^{n-1}}^{t} x^3(s)ds \quad .$$

The system of differential equations

$$\begin{bmatrix} \dot{x}^1(t) \\ \dot{x}^4(t) \\ \dot{y}^1(t) \\ \dot{y}^4(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & -A^*/7 & -A^4 \\ 0 & 0 & -(A^1 + A^*/7) & 0 \\ -B^*/7 & -B^4 & 0 & 0 \\ -(B^1 + B^*/7) & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x^1(t) \\ x^4(t) \\ y^1(t) \\ y^4(t) \end{bmatrix}$$

$$x^1(t^{n-1}) = U_{n1}X_2(t^{n-1}) \quad , \quad x^4(t^{n-1}) = U_{n4}X_2(t^{n-1}) \quad ,$$

$$y^1(t^{n-1}) = V_{n1}Y_2(t^{n-1}) \quad , \quad y^4(t^{n-1}) = V_{n4}Y_2(t^{n-1}) \quad ,$$

can be shown to have the solution

$$\begin{bmatrix} x^1(t) \\ x^4(t) \\ y^1(t) \\ y^4(t) \end{bmatrix} = B \exp\left(\hat{A}(t-t^{n-1})\right) B^{-1} \begin{bmatrix} U_{n1} X_2(t^{n-1}) \\ U_{n4} X_2(t^{n-1}) \\ V_{n1} Y_2(t^{n-1}) \\ V_{n4} Y_2(t^{n-1}) \end{bmatrix}$$

where

$$\hat{A} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & -\lambda_1 & \\ 0 & & & -\lambda_2 \end{bmatrix}$$

$$\lambda_1 = \left\{\frac{-b + [b^2 - 4c]^{\frac{1}{2}}}{2}\right\}^{\frac{1}{2}} \quad , \quad \lambda_2 = \left\{\frac{-b - [b^2 - 4c]^{\frac{1}{2}}}{2}\right\}^{\frac{1}{2}}$$

$$b = -\left(\frac{A^*B^*}{49} + A^4\left(B^1 + \frac{B^*}{7}\right) + B^4\left(A^1 + \frac{A^*}{7}\right)\right)$$

$$c = A^4 B^4 \left(B^1 + \frac{B^*}{7}\right)\left(A^1 + \frac{A^*}{7}\right)$$

$$B = \begin{bmatrix} \dfrac{B^4(A^1 + A^*/7) - \lambda_1^2}{\lambda_1 B^*/7} & \dfrac{B^4(A^1 + A^*/7) - \lambda_2^2}{\lambda_2 B^*/7} & -\dfrac{B^4(A^1 + A^*/7) - \lambda_1^2}{\lambda_1 B^*/7} & -\dfrac{B^4(A^1 + A^*/7) - \lambda_2^2}{\lambda_2 B^*/7} \\ -\dfrac{(A^1 + A^*/7)}{\lambda_1} & -\dfrac{(A^1 + A^*/7)}{\lambda_2} & \dfrac{(A^1 + A^*/7)}{\lambda_1} & \dfrac{(A^1 + A^*/7)}{\lambda_2} \\ 1 & 1 & 1 & 1 \\ -\dfrac{A^*(B^1 + B^*/7)/7}{A^4(B^1 + B^*/7) - \lambda_1^2} & -\dfrac{A^*(B^1 + B^*/7)/7}{A^4(B^1 + B^*/7) - \lambda_2^2} & -\dfrac{A^*(B^1 + B^*/7)/7}{A^4(B^1 + B^*/7) - \lambda_1^2} & -\dfrac{A^*(B^1 + B^*/7)/7}{A^4(B^1 + B^*/7) - \lambda_2^2} \end{bmatrix}$$

$$B^{-1} =$$

$$\left[\begin{matrix}
-\dfrac{\left\{\left(A^{*}B^{*}/49\right)\left(A^{4}\left(B^{1}+B^{*}/7\right)+\lambda_1^2\right)+\left(A^{4}\left(B^{1}+B^{*}/7\right)-\lambda_1^2\right)\left(A^{4}\left(B^{1}+B^{*}/7\right)-\lambda_2^2\right)\right\}}{2A^{*}\lambda_1\left(\lambda_1^2-\lambda_2^2\right)/7}, & -\dfrac{\left\{\left(\left(A^{1}+A^{*}/7\right)B^{4}-\lambda_1\lambda_2\right)\left(\lambda_1+\lambda_2\right)^2+\left(\left(A^{1}+A^{*}/7\right)B^{4}+\lambda_1\lambda_2\right)\left(A^{4}\left(B^{1}+\right.\right.\right.}{2\left(A^{1}+A^{*}/7\right)\left(\lambda_1+\lambda_2\right)\left(\lambda_1^2-\lambda_2^2\right)} \\[3em]
\dfrac{\left(A^{*}B^{*}/49\right)\left(A^{4}\left(B^{1}+B^{*}/7\right)+\lambda_1\lambda_2\right)+\left(A^{4}\left(B^{1}+B^{*}/7\right)-\lambda_1^2\right)\left(A^{4}\left(B^{1}+B^{*}/7\right)-\lambda_2^2\right)}{2A^{*}\lambda_1\left(\lambda_1^2-\lambda_2^2\right)/7}, & \dfrac{2\lambda_2\left(\left(A^{1}+A^{*}/7\right)B^{4}-\lambda_1^2\right)\left(\lambda_1+\lambda_2\right)+\left(\left(A^{1}+A^{*}/7\right)B^{4}+\lambda_1\lambda_2\right)\left(A^{4}\left(B^{1}+\right.\right.}{2\left(A^{1}+A^{*}/7\right)\left(\lambda_1+\lambda_2\right)\left(\lambda_1^2-\lambda_2^2\right)} \\[3em]
-\dfrac{\left(A^{4}\left(B^{1}+B^{*}/7\right)-\lambda_1^2\right)\left(A^{4}\left(B^{1}+B^{*}/7\right)+A^{*}B^{*}/49-\lambda_2^2\right)}{2A^{*}\lambda_1\left(\lambda_1^2-\lambda_2^2\right)/7}, & -\dfrac{\left\{\left(\left(A^{1}+A^{*}/7\right)B^{4}+\lambda_1\lambda_2\right)\left(A^{4}\left(B^{1}+B^{*}/7\right)-\lambda_1^2\right)+A^{*}B^{*}\lambda_1\lambda_2/49\right\}}{2\left(A^{1}+A^{*}/7\right)\left(\lambda_1+\lambda_2\right)\left(\lambda_1^2-\lambda_2^2\right)}, \\[3em]
\dfrac{\left\{\left(A^{*}B^{*}/49\right)\left(A^{4}\left(B^{1}+B^{*}/7\right)-\lambda_1\lambda_2\right)+\left(A^{4}\left(B^{1}+B^{*}/7\right)-\lambda_1^2\right)\left(A^{4}\left(B^{1}+B^{*}/7\right)-\lambda_2^2\right)\right\}}{2A^{*}\lambda_1\left(\lambda_1^2-\lambda_2^2\right)/7}, & \dfrac{\left(\left(A^{1}+A^{*}/7\right)B^{4}+\lambda_1\lambda_2\right)\left(A^{4}\left(B^{1}+B^{*}/7\right)-\lambda_2^2\right)+A^{*}B^{*}\lambda_1\lambda_2/49}{2\left(A^{1}+A^{*}/7\right)\left(\lambda_1+\lambda_2\right)\left(\lambda_1^2-\lambda_2^2\right)},
\end{matrix}\right.$$

$$\frac{\lambda_2\big)^2 + \Big(\big(A^1 + A^*/7\big)B^4 + \lambda_1\lambda_2\Big)\Big(A^4\big(B^1 + B^*/7\big) - \lambda_2^2\Big) + A^*B^*\lambda_1\lambda_2/49\Big\}}{2\big(A^1 + A^*/7\big)\big(\lambda_1 + \lambda_2\big)\big(\lambda_1^2 - \lambda_2^2\big)} \; , \; -\frac{\Big(A^4\big(B^1 + B^*/7\big) - \lambda_1^2\Big)}{2\big(\lambda_1^2 - \lambda_2^2\big)} \; , \; -\frac{\Big(A^4\big(B^1 + B^*/7\big) - \lambda_1^2\Big)\Big(A^4\big(B^1 + B^*/7\big) - \lambda_2^2\Big)}{2A^*\big(B^1 + B^*/7\big)\big(\lambda_1^2 - \lambda_2^2\big)/7}$$

$$\frac{\lambda_2\big) + \Big(\big(A^1 + A^*/7\big)B^4 + \lambda_1\lambda_2\Big)\Big(A^4\big(B^1 + B^*/7\big) - \lambda_2^2\Big) + A^*B^*\lambda_1\lambda_2/49}{2\big(A^1 + A^*/7\big)\big(\lambda_1 + \lambda_2\big)\big(\lambda_1^2 - \lambda_2^2\big)} \; , \; \frac{A^4\big(B^1 + B^*/7\big) - \lambda_2^2}{2\big(\lambda_1^2 - \lambda_2^2\big)} \; , \; \frac{\Big(A^4\big(B^1 + B^*/7\big) - \lambda_1^2\Big)\Big(A^4\big(B^1 + B^*/7\big) - \lambda_2^2\Big)}{2A^*\big(B^1 + B^*/7\big)\big(\lambda_1^2 - \lambda_2^2\big)/7}$$

$$\frac{+ B^*/7\big) - \lambda_1^2\Big) + A^*B^*\lambda_1\lambda_2/49\Big\}}{+ \lambda_2\big)\big(\lambda_1^2 - \lambda_2^2\big)} \; , \; -\frac{\Big(A^4\big(B^1 + B^*/7\big) - \lambda_1^2\Big)}{2\big(\lambda_1^2 - \lambda_2^2\big)} \; , \; -\frac{\Big(A^4\big(B^1 + B^*/7\big) - \lambda_1^2\Big)\Big(A^4\big(B^1 + B^*/7\big) - \lambda_2^2\Big)}{2A^*\big(B^1 + B^*/7\big)\big(\lambda_1^2 - \lambda_2^2\big)/7}$$

$$\frac{+ B^*/7\big) - \lambda_2^2\Big) + A^*B^*\lambda_1\lambda_2/49}{+ \lambda_2\big)\big(\lambda_1^2 - \lambda_2^2\big)} \; , \; \frac{A^4\big(B^1 + B^*/7\big) - \lambda_2^2}{2\big(\lambda_1^2 - \lambda_2^2\big)} \; , \; \frac{\Big(A^4\big(B^1 + B^*/7\big) - \lambda_1^2\Big)\Big(A^4\big(B^1 + B^*/7\big) - \lambda_2^2\Big)}{2A^*\big(B^1 + B^*/7\big)\big(\lambda_1^2 - \lambda_2^2\big)/7} \Bigg]$$

2

D. Comments

Model I was developed without regard for whether it was computationally feasible to solve. The driving consideration in its development was to construct the most aggregate model that still represented the air/ground war in reasonable detail. Its purpose was to provide a benchmark by which subsequent (more computationally tractable) models could be compared.

The difficulty of solving the N-stage game determined by Model I arises from two sources. First, we must optimize over the 14 variables (7 for B and 7 for R) corresponding to the air missions of B and R. While this requirement may not seem difficult at first, the problem confronting us is that there exists no algorithm to solve an infinite game in closed form over N stages. Each such game requires the development and use of techniques taking advantage of the special characteristics of the game. The development of these techniques is hard even when there are as few as four variables, and it becomes progressively harder as the variables increase in number.

The second source of difficulty arises from the form of the payoff and attrition functions. As the solution of the game progresses, we find almost immediately that the value of the game (since it is a function of the payoff and attrition functions) is a complicated function of the strategies of the previous stage. This point will be discussed further in the next chapter, since Model II suffers from this difficulty also.

As a result of the first difficulty, the number of variables, Model II was formulated. In Model II the air missions are aggregated into three primary missions for each side. Once assignment has been made to the primary missions, airplanes assigned to any primary mission are suballocated among the subtasks of that mission.

III MODEL II

In the following, all assumptions of Model I apply unless we indicate otherwise. Where applicable the same notation is used.

Recall from Figure 1 the original seven-air-mission aggregated air-and-ground-war model. In Figure 2 the battlefield of the seven-mission model has been divided into three regions: T_B, the region defended by B and attacked by R; T_R, the region defended by R and attacked by B; and T_{BR}, the region that may be both defended and attacked by B and R. Thus T_B contains the B logistic pipeline and B air base, T_R contains the R logistic pipeline and R air base, and T_{BR} contains the B ground force, the R ground force and the air space above the ground forces.

Under this aggregation scheme B and R allocate airplanes to the three regions at times t^{n-1}, $1 \le n \le N$. An airplane allocated by B(R) to region $T_B(T_R)$ will ultimately be suballocated to B(R)'s AD vs AN or AD vs I mission; an airplane allocated by B(R) to region $T_R(T_B)$ will ultimately be suballocated to B(R)'s AN or I mission; an airplane allocated by B(R) to region T_{BR} will ultimately be suballocated to B(R)'s AS or CAS or AD vs CAS mission. Let α_n (β_n) be the fraction of B(R) airplanes allocated to $T_R(T_B)$ at time t^{n-1} that is ultimately allocated to AN. Let δ_n (η_n) be the fraction of B(R) airplanes allocated to T_{BR} at time t^{n-1} that is ultimately allocated to CAS.

We assume now that B(R)'s air allocation to T_q $(q = B,R,BR)$ is exposed to attrition from R(B)'s air allocation to T_q as a unit, and thus independent of the ultimate assignment of airplanes to air missions which results from the suballocation within region T_q. Furthermore, at each time t^{n-1}, $1 \le n \le N$, the air to ground fire of those B(R) airplanes

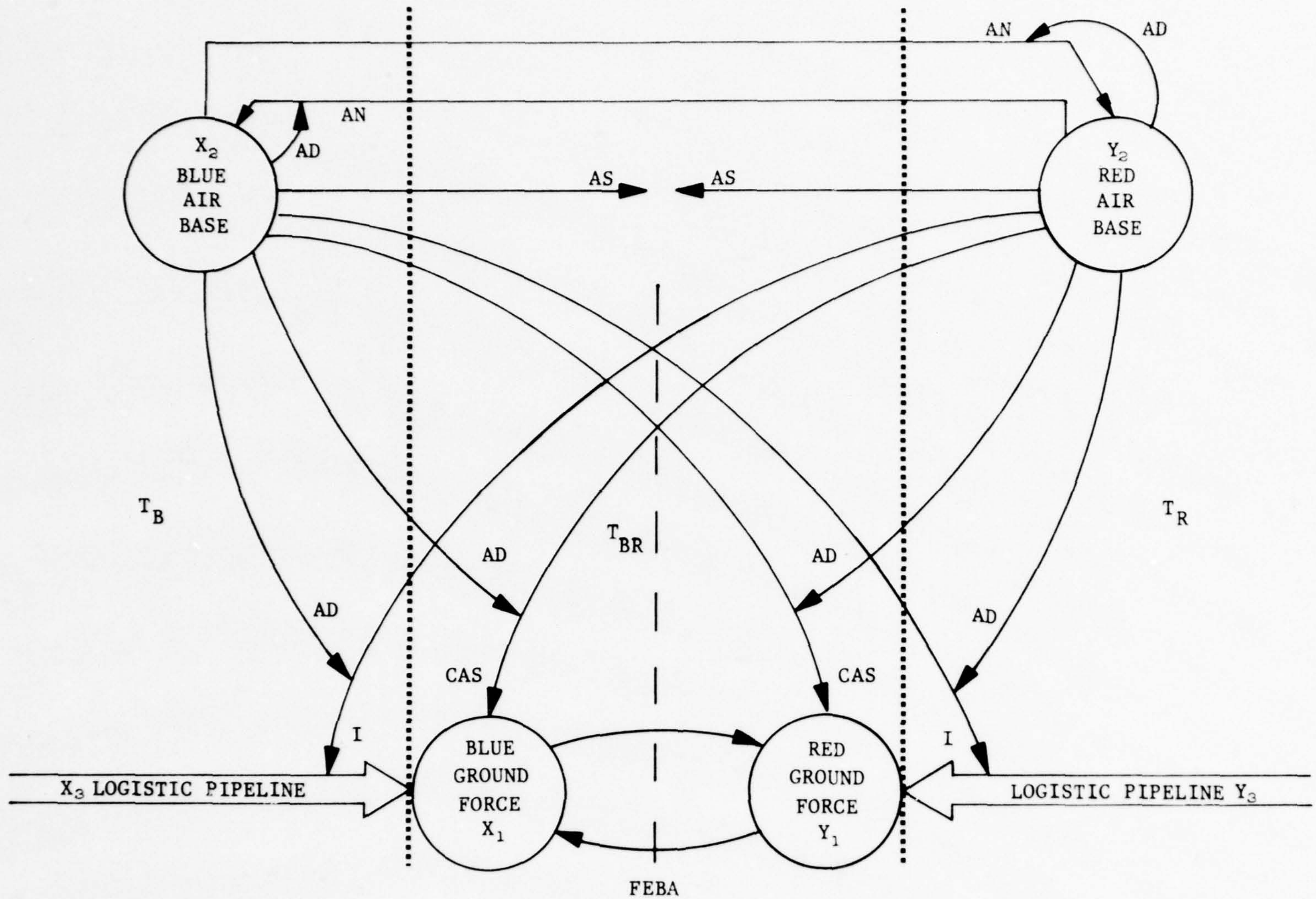

FIGURE 2. THE DIVISION OF THE BATTLEFIELD INTO T_B, T_R, AND T_{BR}

ultimately assigned to AN is assumed to be uniformly allocated among R(B)'s missions to T_B, T_R, T_{BR}.

An air allocation for B(R) in stage n is a vector $\zeta_n(\gamma_n) \in \mathbb{R}^3$ where $\zeta_n = (\zeta_{ni})$, $\gamma_n = (\gamma_{ni})$, $\zeta_{ni}, \gamma_{ni} \geq 0$, and $\sum_i \zeta_{ni} = \sum_i \gamma_{ni} = 1$. ζ_{n1}, ζ_{n2}, and ζ_{n3} (γ_{n1}, γ_{n2}, and γ_{n3}) are the fractions of available airplanes at time t^{n-1} assigned by B(R) to T_B, T_R, and T_{BR}, respectively.

$X_1(t)$, $X_2(t)$, $X_3(t)$, $Y_1(t)$, $Y_2(t)$, and $Y_3(t)$ are as previously defined. $x_B(t)\big(y_B(t)\big)$ is the order of battle of B(R)'s T_B mission (= all airplanes allocated to T_B) at time t. $x_R(t)\big(y_R(t)\big)$ is the order of battle of B(R)'s T_R mission. $x_{BR}(t)\big(y_{BR}(t)\big)$ is the order of battle of B(R)'s T_{BR} mission.

Figure 3 gives a graphic representation of the aggregation into the regions T_B, T_R, and T_{BR}, and the subsequent suballocation within each of these regions.

A. Differential Equations

The following system of differential equations arises for stage n, $1 \leq n \leq N$. Throughout $t \in [t^{n-1}, t^n]$.

1. Ground War

$$\varphi_n = \int_{t^{n-1}}^{t^n} \varphi(t)dt$$

$$\begin{bmatrix} \dot{X}_1(t) \\ \dot{Y}_1(t) \end{bmatrix} = \begin{bmatrix} 0 & -A_1(n, \zeta_{n-1\,2}, \gamma_{n-1\,2}) \\ -B_1(n, \zeta_{n-1\,1}, \gamma_{n-1\,1}) & 0 \end{bmatrix} \begin{bmatrix} X_1(t) \\ Y_1(t) \end{bmatrix} - \begin{bmatrix} a_1(n, \gamma_{n3}) \\ b_1(n, \zeta_{n3}) \end{bmatrix}$$

$$X_1(t),\ Y_1(t) \geq 0,\ X_1(0) \equiv X_1,\ Y_1(0) \equiv Y_1 \quad .$$

2. Logistic Pipelines

$$\dot{X}_3(t) = \xi_n^B - (1-\beta_n)A^{**}y_B(t) \quad ,$$

$$\dot{Y}_3(t) = \xi_n^R - (1-\alpha_n)B^{**}x_R(t) \quad ,$$

$$X_3(t),\ Y_3(t) \geq 0 \quad .$$

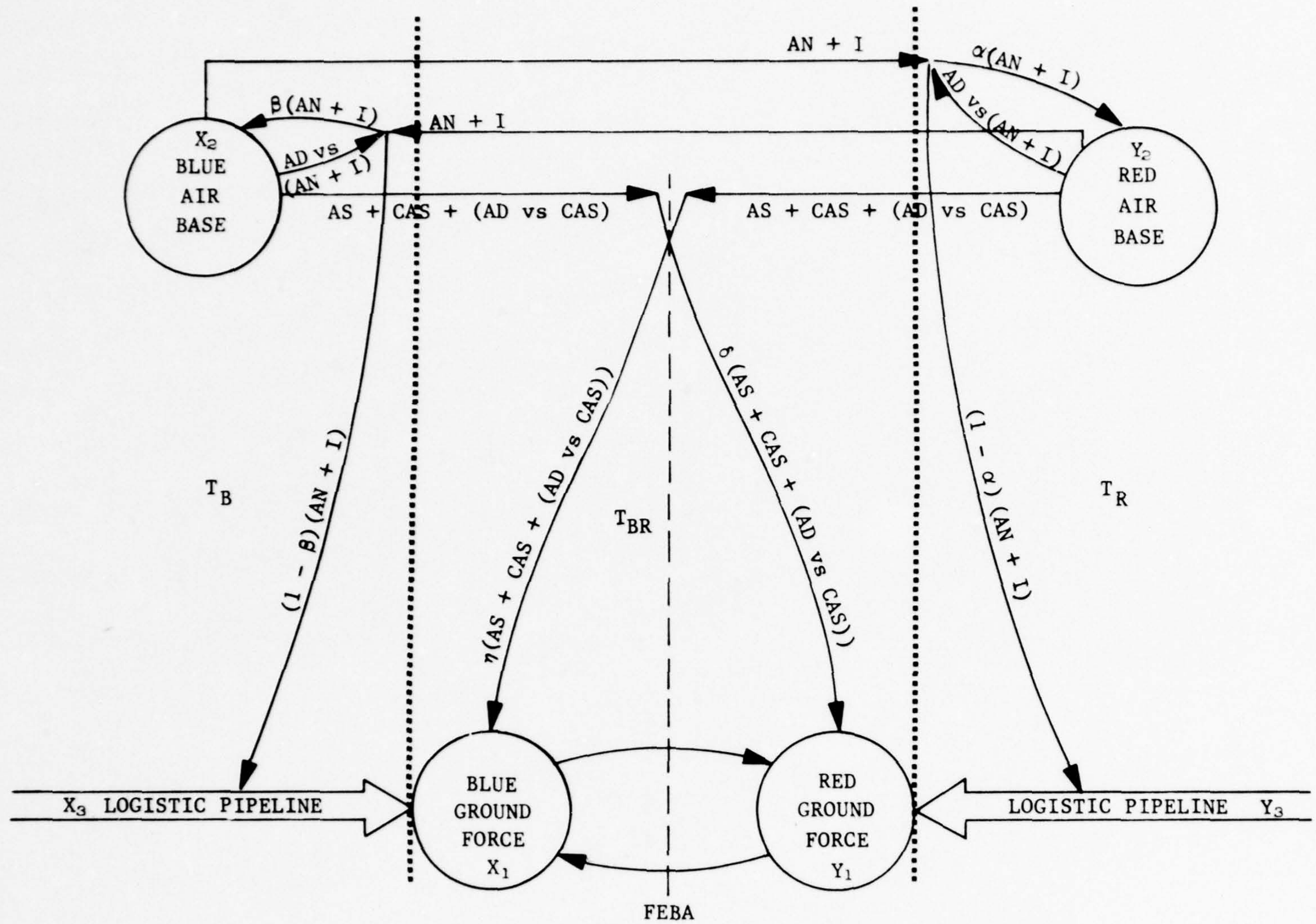

FIGURE 3. GRAPHIC REPRESENTATION OF AGGREGATION INTO THE 3 REGIONS T_B, T_R, T_{BR}. α, β, δ, η MUST BE DETERMINED IN THE SUBALLOCATION.

3. Air War

T_B and T_R

$$\begin{bmatrix} \dot{x}_R(t) \\ \dot{x}_B(t) \\ \dot{y}_B(t) \\ \dot{y}_R(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & -A^*\beta_n/3 & -A_R \\ 0 & 0 & -(A_B + A^*\beta_n/3) & 0 \\ -B^*\alpha_n/3 & -B_B & 0 & 0 \\ -(B_R + B^*\alpha_n/3) & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_R(t) \\ x_B(t) \\ y_B(t) \\ y_R(t) \end{bmatrix}$$

$$x_R(t^{n-1}) = \zeta_{n2}X_2(t^{n-1}),\ x_B(t^{n-1}) = \zeta_{n1}X_2(t^{n-1}),$$

$$y_B(t^{n-1}) = \gamma_{n1}Y_2(t^{n-1}),\ y_R(t^{n-1}) = \gamma_{n2}Y_2(t^{n-1}),$$

$$x_R(t),\ x_B(t),\ y_B(t),\ y_R(t) \geq 0 \quad .$$

T_{BR}

$$\begin{bmatrix} \dot{x}_{BR}(t) \\ \dot{y}_{BR}(t) \end{bmatrix} = \begin{bmatrix} 0 & -A_{BR} \\ -B_{BR} & 0 \end{bmatrix} \begin{bmatrix} x_{BR}(t) \\ y_{BR}(t) \end{bmatrix} - \begin{bmatrix} A^*\beta_n y_B(t)/3 \\ B^*\alpha_n x_R(t)/3 \end{bmatrix}$$

$$x_{BR}(t^{n-1}) = \zeta_{n3}X_2(t^{n-1}),\ y_{BR}(t^{n-1}) = \gamma_{n3}Y_2(t^{n-1}),$$

$$x_{BR}(t),\ y_{BR}(t) \geq 0 \quad .$$

4. Modeling $A_1(n,\zeta_{n-1\,2},\gamma_{n-1\,2})$, $B_1(n,\zeta_{n-1\,1},\gamma_{n-1\,1})$

We assume

$$A_1(n,\zeta_{n-1\,2},\gamma_{n-1\,2}) = f_n^R(\zeta_{n-1\,2},\gamma_{n-1\,2},\rho_R(t^{n-1}))A_1'(n)$$

$$B_1(n,\zeta_{n-1\,1},\gamma_{n-1\,1}) = f_n^B(\zeta_{n-1\,1},\gamma_{n-1\,1},\rho_B(t^{n-1}))B_1'(n)$$

where $\rho_q(t^{n-1})$, $A_1'(n)$, $B_1'(n)$ are as previously defined.

Let

$$f_n^B(\zeta_{n-1\,1},\gamma_{n-1\,1},\rho_B(t^{n-1})) = \frac{\rho_B(t^{n-1}) \wedge xX_3(t^{n-1})}{\rho_B(t^{n-1})} ,$$

$$f_n^R(\zeta_{n-1\,2},\gamma_{n-1\,2},\rho_R(t^{n-1})) = \frac{\rho_R(t^{n-1}) \wedge yY_3(t^{n-1})}{\rho_R(t^{n-1})} .$$

<u>Modeling $a_1(n,\gamma_{n3})$, $b_1(n,\zeta_{n3})$</u>

$$a_1(n,\gamma_{n3}) = a_1'(n)\gamma_{n3}\eta_n Y_2(t^{n-1})$$

$$b_1(n,\zeta_{n3}) = b_1'(n)\zeta_{n3}\delta_n X_2(t^{n-1}) .$$

B. The Formulated Game

B(R) picks $\zeta = (\zeta_n)\left(\gamma = (\gamma_n)\right)$ to max(min) the payoff function

$$\left[X_1(T) - Y_1(T)\right]$$

so that for $1 \le n \le N$, $t^{n-1} \le t \le t^n$

$$\begin{bmatrix} \dot{X}_1(t) \\ \\ \dot{Y}_1(t) \end{bmatrix} = \begin{bmatrix} 0 & -\dfrac{\left(\rho_R(t^{n-1}) \wedge yY_3(t^{n-1})\right)}{\rho_R(t^{n-1})} A'(n) \\ \\ -\dfrac{\left(\rho_B(t^{n-1}) \wedge xX_3(t^{n-1})\right)}{\rho_B(t^{n-1})} B'(n) & 0 \end{bmatrix} \begin{bmatrix} X_1(t) \\ \\ Y_1(t) \end{bmatrix}$$

$$- \begin{bmatrix} a_1'(n)\gamma_{n3}\eta_n Y_2(t^{n-1}) \\ b_1'(n)\zeta_{n3}\delta_n X_2(t^{n-1}) \end{bmatrix} ;$$

$$\dot{X}_3(t) = \xi_n^B - (1-\beta_n)A^{**}y_B(t),$$

$$\dot{Y}_3(t) = \xi_n^R - (1-\alpha_n)B^{**}x_R(t);$$

$$\begin{bmatrix} \dot{x}_R(t) \\ \dot{x}_B(t) \\ \dot{y}_B(t) \\ \dot{y}_R(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & -A^*\beta_n/3 & -A_R \\ 0 & 0 & -(A_B + A^*\beta_n/3) & 0 \\ -B^*\alpha_n/3 & -B_B & 0 & 0 \\ -(B_R + B^*\alpha_n/3) & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_R(t) \\ x_B(t) \\ y_B(t) \\ y_R(t) \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_{BR}(t) \\ \dot{y}_{BR}(t) \end{bmatrix} = \begin{bmatrix} 0 & -A_{BR} \\ -B_{BR} & 0 \end{bmatrix} \begin{bmatrix} x_{BR}(t) \\ y_{BR}(t) \end{bmatrix} - \begin{bmatrix} A^*\beta_n y_B(t)/3 \\ B^*\alpha_n x_R(t)/3 \end{bmatrix}$$

$$X_1(t),\ Y_1(t),\ X_3(t),\ Y_3(t),\ x_q(t),\ y_q(t) \ge 0 \quad (q = B,R,BR);$$

$$x_R(t^{n-1}) = \zeta_{n2}X_2(t^{n-1}),\ x_B(t^{n-1}) = \zeta_{n1}X_2(t^{n-1}),$$

$$y_B(t^{n-1}) = \gamma_{n1}Y_2(t^{n-1}),\ y_R(t^{n-1}) = \gamma_{n2}Y_2(t^{n-1}),$$

$$x_{BR}(t^{n-1}) = \zeta_{n3}X_2(t^{n-1}),\ y_{BR}(t^{n-1}) = \gamma_{n3}Y_2(t^{n-1});$$

and $X_i(0) \equiv X_i$, $Y_i(0) \equiv Y_i$.

C. The Solution of the Differential Equations

Throughout $t \geq t^{n-1}$. The system

$$\begin{bmatrix}\dot{X}_1(t)\\ \\ \dot{Y}_1(t)\end{bmatrix} = \begin{bmatrix} 0 & \dfrac{-\left(\rho_R(t^{n-1}) \wedge yY_3(t^{n-1})\right)A'(n)}{\rho_R(t^{n-1})} \\ \\ \dfrac{-\left(\rho_B(t^{n-1}) \wedge xX_3(t^{n-1})\right)B'(n)}{\rho_B(t^{n-1})} & 0 \end{bmatrix}\begin{bmatrix}X_1(t)\\ \\ Y_1(t)\end{bmatrix}$$

$$+ \begin{bmatrix} -\, a_1'(n)\gamma_{n3}\eta_n Y_2(t^{n-1}) \\ \\ -\, b_1'(n)\zeta_{n3}\delta_n X_2(t^{n-1}) \end{bmatrix}$$

$X_1(t^{n-1}),\ Y_1(t^{n-1})$

has the solution

$$X_1(t) = \left(X_1(t^{n-1}) + \frac{\left(b_1'(n)\zeta_{n3}\delta_n X_2(t^{n-1})\right)\rho_B(t^{n-1})}{\left(\rho_B(t^{n-1}) \wedge xX_3(t^{n-1})\right)B'(n)}\right)$$

$$\times \cosh\left(\left[\frac{\left(\rho_R(t^{n-1}) \wedge yY_3(t^{n-1})\right)A'(n)\left(\rho_B(t^{n-1}) \wedge xX_3(t^{n-1})\right)B'(n)}{\rho_R(t^{n-1})\rho_B(t^{n-1})}\right]^{\frac{1}{2}}(t-t^{n-1})\right)$$

$$-\left(Y_1(t^{n-1}\left[\frac{\left(\rho_R(t^{n-1}) \wedge yY_3(t^{n-1})\right)A'(n)\rho_B(t^{n-1})}{\rho_R(t^{n-1})\left(\rho_B(t^{n-1}) \wedge xX_3(t^{n-1})\right)B'(n)}\right]^{\frac{1}{2}}\right.$$

$$\left. + \frac{a_1'(n)\gamma_{n3}\eta_n Y_2(t^{n-1})}{\left[\dfrac{\left(\rho_R(t^{n-1}) \wedge yY_3(t^{n-1})\right)A'(n)\left(\rho_B(t^{n-1}) \wedge xX_3(t^{n-1})\right)B'(n)}{\rho_R(t^{n-1})\rho_B(t^{n-1})}\right]^{\frac{1}{2}}}\right)$$

$$\times \sinh\left(\left[\frac{\left(\rho_R(t^{n-1}) \wedge yY_3(t^{n-1})\right)A'(n)\left(\rho_B(t^{n-1}) \wedge xX_3(t^{n-1})\right)B'(n)}{\rho_R(t^{n-1})\rho_B(t^{n-1})}\right]^{\frac{1}{2}}(t-t^{n-1})\right)$$

$$-\ \frac{b_1'(n)\zeta_{n3}\delta_n X_2(t^{n-1})\rho_B(t^{n-1})}{\left(\rho_B(t^{n-1}) \wedge xX_3(t^{n-1})\right)B'(n)}\ ,$$

and

$$Y_1(t) = \left(Y_1(t^{n-1}) + \frac{a_1'(n)\gamma_{n3}\eta_n Y_2(t^{n-1})\rho_R(t^{n-1})}{\left(\rho_R(t^{n-1}) \wedge yY_3(t^{n-1})\right)A'(n)}\right)$$

$$\times \cosh\left(\left[\frac{\left(\rho_R(t^{n-1}) \wedge yY_3(t^{n-1})\right)A'(n)\left(\rho_B(t^{n-1}) \wedge xX_3(t^{n-1})\right)B'(n)}{\rho_R(t^{n-1})\rho_B(t^{n-1})}\right]^{\frac{1}{2}}(t-t^{n-1})\right)$$

$$- \left(X_1(t^{n-1}) \left[\frac{\left(\rho_B(t^{n-1}) \wedge xX_3(t^{n-1})\right)B'(n)\rho_R(t^{n-1})}{\rho_B(t^{n-1})\left(\rho_R(t^{n-1}) \wedge yY_3(t^{n-1})\right)A'(n)}\right]^{\frac{1}{2}}\right.$$

$$\left. + \frac{b_1'(n)\zeta_{n3}\delta_n X_2(t^{n-1})}{\left[\frac{\left(\rho_R(t^{n-1}) \wedge yY_3(t^{n-1})\right)A'(n)\left(\rho_B(t^{n-1}) \wedge xX_3(t^{n-1})\right)B'(n)}{\rho_R(t^{n-1})\rho_B(t^{n-1})}\right]^{\frac{1}{2}}}\right)$$

$$\times \sinh\left(\left[\frac{\left(\rho_R(t^{n-1}) \wedge yY_3(t^{n-1})\right)A'(n)\left(\rho_B(t^{n-1}) \wedge xX_3(t^{n-1})\right)B'(n)}{\rho_R(t^{n-1})\rho_B(t^{n-1})}\right]^{\frac{1}{2}}(t-t^{n-1})\right)$$

$$- \frac{a_1'(n)\gamma_{n3}\eta_n Y_2(t^{n-1})\rho_R(t^{n-1})}{\left(\rho_R(t^{n-1}) \wedge yY_3(t^{n-1})\right)A'(n)} \quad .$$

The differential equation

$$\dot{X}_3(t) = \xi_n^B - (1-\beta_n)A^{**}y_B(t),\ X_3(t^{n-1})$$

has the solution

$$X_3(t) = X_3(t^{n-1}) + \xi_n^B\ (t-t^{n-1}) - (1-\beta_n)A^{**}\int_{t^{n-1}}^{t} y_B(s)ds \quad .$$

The differential equation

$$\dot{Y}_3(t) = \xi_n^R - (1-\alpha_n)B^{**}x_R(t),\ Y_3(t^{n-1})$$

has the solution

$$Y_3(t) = Y_3(t^{n-1}) + \xi_n^R\ (t-t^{n-1}) - (1-\alpha_n)B^{**}\int_{t^{n-1}}^{t} x_R(s)ds \quad .$$

Routine calculations show that the system of differential equations

$$\begin{bmatrix} \dot{x}_R(t) \\ \dot{x}_B(t) \\ \dot{y}_B(t) \\ \dot{y}_R(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & -A^*\beta_n/3 & -A_R \\ 0 & 0 & -(A_B + A^*\beta_n/3) & 0 \\ -B^*\alpha_n/3 & -B_B & 0 & 0 \\ -(B_R + B^*\alpha_n/3) & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_R(t) \\ x_B(t) \\ y_B(t) \\ y_R(t) \end{bmatrix} \tag{1}$$

$$x_R(t^{n-1}) = \zeta_{n2}X_2(t^{n-1}) \quad , \quad x_B(t^{n-1}) = \zeta_{n1}X_2(t^{n-1}) \quad ,$$

$$y_B(t^{n-1}) = \gamma_{n1}Y_2(t^{n-1}) \quad , \quad y_R(t^{n-1}) = \gamma_{n2}Y_2(t^{n-1})$$

has the solution

$$\begin{bmatrix} x_R(t) \\ x_B(t) \\ y_B(t) \\ y_R(t) \end{bmatrix} = B \exp\left(\hat{A}(t-t^{n-1})\right) B^{-1} \begin{bmatrix} \zeta_{n2}X_2(t^{n-1}) \\ \zeta_{n1}X_2(t^{n-1}) \\ \gamma_{n1}Y_2(t^{n-1}) \\ \gamma_{n2}Y_2(t^{n-1}) \end{bmatrix}$$

where

$$\hat{A} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & -\lambda_1 & \\ 0 & & & -\lambda_2 \end{bmatrix} ,$$

$$\lambda_1 = \left\{\frac{-b + [b^2 - 4c]^{\frac{1}{2}}}{2}\right\}^{\frac{1}{2}} \quad , \quad \lambda_2 = \left\{\frac{-b - [b^2 - 4c]^{\frac{1}{2}}}{2}\right\}^{\frac{1}{2}} \quad ,$$

$$b = -\left\{(A_B + A^*\beta_n/3)B_B + A^*\beta_n B^*\alpha_n/9 + A_R(B_R + B^*\alpha_n/3)\right\} \quad ,$$

$$c = A_R B_B (A_B + A^*\beta_n/3)(B_R + B^*\alpha_n/3) \quad ,$$

$$
B = \begin{bmatrix}
\dfrac{\lambda_1 A^*\beta_n/3}{A_R\left(B_R + B^*\alpha_n/3\right) - \lambda_1^2} & \dfrac{\lambda_2 A^*\beta_n/3}{A_R\left(B_R + B^*\alpha_n/3\right) - \lambda_2^2} & -\dfrac{\lambda_1 A^*\beta_n/3}{A_R\left(B_R + B^*\alpha_n/3\right) - \lambda_1^2} & -\dfrac{\lambda_2 A^*\beta_n/3}{A_R\left(B_R + B^*\alpha_n/3\right) - \lambda_2^2} \\
-\dfrac{\left(A_B + A^*\beta_n/3\right)}{\lambda_1} & -\dfrac{\left(A_B + A^*\beta_n/3\right)}{\lambda_2} & \dfrac{A_B + A^*\beta_n/3}{\lambda_1} & \dfrac{A_B + A^*\beta_n/3}{\lambda_2} \\
1 & 1 & 1 & 1 \\
-\dfrac{\left(A^*\beta_n/3\right)\left(B_R + B^*\alpha_n/3\right)}{A_R\left(B_R + B^*\alpha_n/3\right) - \lambda_1^2} & -\dfrac{\left(A^*\beta_n/3\right)\left(B_R + B^*\alpha_n/3\right)}{A_R\left(B_R + B^*\alpha_n/3\right) - \lambda_2^2} & -\dfrac{\left(A^*\beta_n/3\right)\left(B_R + B^*\alpha_n/3\right)}{A_R\left(B_R + B^*\alpha_n/3\right) - \lambda_1^2} & -\dfrac{\left(A^*\beta_n/3\right)\left(B_R + B^*\alpha_n/3\right)}{A_R\left(B_R + B^*\alpha_n/3\right) - \lambda_2^2}
\end{bmatrix}
$$

and B^{-1} =

$$
\begin{bmatrix}
-\dfrac{B^*\alpha_n\lambda_1}{6\left(\lambda_1^2-\lambda_2^2\right)} & -\dfrac{\lambda_1\left[\left(A_B+A^*\beta_n/3\right)B_B-\lambda_2^2\right]}{2\left(A_B+A^*\beta_n/3\right)\left(\lambda_1^2-\lambda_2^2\right)} & -\dfrac{A_R\left(B_R+B^*\alpha_n/3\right)-\lambda_1^2}{2\left(\lambda_1^2-\lambda_2^2\right)} & \dfrac{A_RB^*\alpha_n}{6\left(\lambda_1^2-\lambda_2^2\right)} \\
\dfrac{B^*\alpha_n\lambda_2}{6\left(\lambda_1^2-\lambda_2^2\right)} & \dfrac{\lambda_2\left[\left(A_B+A^*\beta_n/3\right)B_B-\lambda_1^2\right]}{2\left(A_B+A^*\beta_n/3\right)\left(\lambda_1^2-\lambda_2^2\right)} & \dfrac{A_R\left(B_R+B^*\alpha_n/3\right)-\lambda_2^2}{2\left(\lambda_1^2-\lambda_2^2\right)} & -\dfrac{A_RB^*\alpha_n}{6\left(\lambda_1^2-\lambda_2^2\right)} \\
\dfrac{B^*\alpha_n\lambda_1}{6\left(\lambda_1^2-\lambda_2^2\right)} & \dfrac{\lambda_1\left[\left(A_B+A^*\beta_n/3\right)B_B-\lambda_2^2\right]}{2\left(A_B+A^*\beta_n/3\right)\left(\lambda_1^2-\lambda_2^2\right)} & -\dfrac{A_R\left(B_R+B^*\alpha_n/3\right)-\lambda_1^2}{2\left(\lambda_1^2-\lambda_2^2\right)} & \dfrac{A_RB^*\alpha_n}{6\left(\lambda_1^2-\lambda_2^2\right)} \\
-\dfrac{B^*\alpha_n\lambda_2}{6\left(\lambda_1^2-\lambda_2^2\right)} & -\dfrac{\lambda_2\left[\left(A_B+A^*\beta_n/3\right)B_B-\lambda_1^2\right]}{2\left(A_B+A^*\beta_n/3\right)\left(\lambda_1^2-\lambda_2^2\right)} & \dfrac{A_R\left(B_R+B^*\alpha_n/3\right)-\lambda_2^2}{2\left(\lambda_1^2-\lambda_2^2\right)} & -\dfrac{A_RB^*\alpha_n}{6\left(\lambda_1^2-\lambda_2^2\right)}
\end{bmatrix}
$$

D. Comments

We make the following definitions:

$$t_B^n = \inf\left\{t \geq t^{n-1} : X_1(t) = 0 \quad , \quad Y_1(t) > 0\right\}$$

$$t_R^n = \inf\left\{t \geq t^{n-1} : Y_1(t) = 0 \quad , \quad X_1(t) > 0\right\}$$

$$\left(t_B^n = \infty \quad \text{if} \quad X_1(t) > 0 \qquad t \geq t^{n-1}\right.$$

and

$$\left. t_R^n = \infty \quad \text{if} \quad Y_1(t) > 0 \qquad t \geq t^{n-1}\right) \quad ;$$

$$t_q^n(s) = \left(s \wedge t_q^n\right) - t^{n-1} \qquad q = B,R \quad ;$$

$$\tau_B^n = \inf\left\{t \geq t_R^n : X_1(t) = 0\right\} \quad ,$$

$$\tau_R^n = \inf\left\{t \geq t_B^n : Y_1(t) = 0\right\} \quad ,$$

$$\left(\tau_B^n = \infty \quad \text{if} \quad X_1(t) > 0 \quad , \quad t \geq t_R^n \right. \quad ,$$

$$\left. \tau_R^n = \infty \quad \text{if} \quad Y_1(t) > 0 \quad , \quad t \geq t_B^n\right) \quad .$$

Incorporating the restriction that the order of battle of the various combat forces must be nonnegative, the ground force levels at stage n are given by:

$$\hat{X}_1(t) = \begin{cases} X_1(t) \quad , & \text{if} \quad t \leq t_B^n \wedge t_R^n \\ X_1\left(t_R^n\right) - \gamma_{n3} a_1'(n) \eta_n Y_2(t^{n-1})\left(t - t_R^n\right), & \text{if} \quad t_R^n < t_B^n \quad \text{and} \\ & \quad t_R^n < t \leq \tau_B^n \\ 0 \quad , & \text{if} \quad t_B^n < t_R^n \quad \text{and} \quad t_B^n < t \\ & \text{or} \quad t_R^n < t_B^n \quad \text{and} \quad \tau_B^n < t \quad ; \end{cases}$$

$$\hat{Y}_1(t) = \begin{cases} Y_1(t) \quad , & \text{if} \quad t \le t_B^n \wedge t_R^n \\ Y_1\left(t_B^n\right) - \zeta_{n3} b_1'(n) \delta_n X_2(t^{n-1})\left(t - t_B^n\right), & \text{if} \quad t_B^n < t_R^n \quad \text{and} \\ & \quad t_B^n < t \le \tau_R^n \\ 0 \quad , & \text{if} \quad t_R^n < t_B^n \quad \text{and} \quad t_R^n < t \\ & \text{or} \quad t_B^n < t_R^n \quad \text{and} \quad \tau_R^n < t \quad . \end{cases}$$

The payoff at stage n is therefore:

$$\hat{X}_1(t^n) - \hat{Y}_1(t^n) = \begin{cases} X_1(t^n) - Y_1(t^n) \quad , & \text{if} \quad t^n \le t_B^n \wedge t_R^n \\ X_1\left(t_R^n\right) - \gamma_{n3} a_1'(n) \eta_n Y_2(t^{n-1})\left(t^n - t_R^n\right), & \text{if} \quad t_R^n < t_B^n \quad \text{and} \\ & \quad t_R^n < t^n \le \tau_B^n \\ -Y_1\left(t_B^n\right) + \zeta_{n3} b_1'(n) \delta_n X_2(t^{n-1})\left(t^n - t_B^n\right), & \text{if} \quad t_B^n < t_R^n \quad \text{and} \\ & \quad t_B^n < t^n \le \tau_R^n \\ 0 \quad , & \text{if otherwise} \quad . \end{cases}$$

It can be shown that

$$t_B^n = t^{n-1} + (a_n b_n)^{-\frac{1}{2}} \ln\left(\frac{b_n\left(X_1(t^{n-1}) + \frac{b_{1n}\zeta_{n3}}{b_n}\right) + (a_n b_n)^{\frac{1}{2}}\left(Y_1(t^{n-1}) + \frac{a_{1n}\gamma_{n3}}{a_n}\right)}{b_{1n}\zeta_{n3} + \left((b_{1n}\zeta_{n3})^2 - \alpha_1^2 b_n\right)^{\frac{1}{2}}}\right)$$

when $\quad b_n^{\frac{1}{2}}\left(X_1(t^{n-1}) + \frac{b_{1n}\zeta_{n3}}{b_n}\right) > a_n^{\frac{1}{2}}\left(Y_1(t^{n-1}) + \frac{a_{1n}\gamma_{n3}}{a_n}\right)$

implies $$b_{1n}\zeta_{n3}/b_n^{\frac{1}{2}} \ge \alpha_1 \quad ,$$

$$t_B^n = \infty \text{ otherwise} \quad ;$$

and

$$t_R^n = t^{n-1} + (a_n b_n)^{-\frac{1}{2}} \ln\left(\frac{(a_n b_n)^{\frac{1}{2}}\left(X_1(t^{n-1}) + \frac{b_{1n}\zeta_{n3}}{b_n}\right) + a_n\left(Y_1(t^{n-1}) + \frac{a_{1n}\gamma_{n3}}{a_n}\right)}{a_{1n}\gamma_{n3} + \left((a_{1n}\gamma_{n3})^2 + \alpha_1^2 a_n\right)^{\frac{1}{2}}}\right)$$

when $$b_n^{\frac{1}{2}}\left(X_1(t^{n-1}) + \frac{b_{1n}\zeta_{n3}}{b_n}\right) < a_n^{\frac{1}{2}}\left(Y_1(t^{n-1}) + \frac{a_{1n}\gamma_{n3}}{a_n}\right)$$

implies $$a_{1n}\gamma_{n3}/a_n^{\frac{1}{2}} \ge \alpha_2 \quad ,$$

$$t_R^n = \infty \text{ otherwise} \quad ;$$

where

$$\alpha_1^2 = b_n\left(X_1(t^{n-1}) + \frac{b_{1n}\zeta_{n3}}{b_n}\right)^2 - a_n\left(Y_1(t^{n-1}) + \frac{a_{1n}\gamma_{n3}}{a_n}\right)^2 = -\alpha_2^2$$

$$a_n = A_1(n, \zeta_{n-12}, \gamma_{n-12})$$

$$= A_1'(n)\,\frac{\left(\rho_R(t^{n-1}) \wedge yY_3(t^{n-1})\right)}{\rho_R(t^{n-1})}$$

$$b_n = B_1(n, \zeta_{n-11}, \gamma_{n-11})$$

$$= B_1'(n)\,\frac{\left(\rho_B(t^{n-1}) \wedge xX_3(t^{n-1})\right)}{\rho_B(t^{n-1})}$$

$$a_{1n} = a_1'(n)\eta_n Y_2(t^{n-1})$$

$$b_{1n} = b_1'(n)\delta_n X_2(t^{n-1}) \quad .$$

We verify that $\zeta_{N3} = 1$, $\gamma_{N3} = 1$ are optimal strategies for the game at stage N. Suppose $t^{N-1} \le t \le t_B^N \wedge t_R^N$, writing out $\hat{X}_1(t)$ and $\hat{Y}_1(t)$, where functional dependence has been suppressed except with respect to ζ and γ, we have

$$\begin{aligned}\hat{X}_1(t) = X_1(t^{N-1}) \cosh\left((a_N b_N)^{\frac{1}{2}}(t-t^{N-1})\right) \\ - Y_1(t^{N-1}) \left(\frac{a_N}{b_N}\right)^{\frac{1}{2}} \sinh\left((a_N b_N)^{\frac{1}{2}}(t-t^{N-1})\right) \\ + \zeta_{N3} \frac{b_{1N}}{b_N}\left(\cosh\left((a_N b_N)^{\frac{1}{2}}(t-t^{N-1})\right) - 1\right) \\ - \gamma_{N3} \frac{a_{1N}}{(a_N b_N)^{\frac{1}{2}}} \sinh\left((a_N b_N)^{\frac{1}{2}}(t-t^{N-1})\right)\end{aligned}$$

$$\begin{aligned}\hat{Y}_1(t) = Y_1(t^{N-1}) \cosh\left((a_N b_N)^{\frac{1}{2}}(t-t^{N-1})\right) \\ - X_1(t^{N-1}) \left(\frac{b_N}{a_N}\right)^{\frac{1}{2}} \sinh\left((a_N b_N)^{\frac{1}{2}}(t-t^{N-1})\right) \\ + \gamma_{N3} \frac{a_{1N}}{a_N}\left(\cosh\left((a_N b_N)^{\frac{1}{2}}(t-t^{N-1})\right) - 1\right) \\ - \zeta_{N3} \frac{b_{1N}}{(a_N b_N)^{\frac{1}{2}}} \sinh\left((a_N b_N)^{\frac{1}{2}}(t-t^{N-1})\right) .\end{aligned}$$

Since a_N, a_{1N}, b_N, $b_{1N} \ge 0$ and $\cosh(u) \ge 1$ it is clear that $\hat{X}_1(t)$ is nondecreasing in ζ_{N3} and nonincreasing in γ_{N3}, while $\hat{Y}_1(t)$ is nondecreasing in γ_{N3} and nonincreasing in ζ_{N3}, all this for any fixed t. Thus $t_B^N\left(t_R^N\right)$ is nondecreasing in $\zeta_{N3}(\gamma_{N3})$ and nonincreasing in $\gamma_{N3}(\zeta_{N3})$. Suppose $t_R^N < t_B^N$ and $t_R^N < t \le \tau_B^N$, then

$$\hat{X}_1(t) = X_1(t^{N-1}) \cosh\left((a_N b_N)^{\frac{1}{2}}\left(t_R^N - t^{N-1}\right)\right)$$

$$- Y_1(t^{N-1})\left(\frac{a_N}{b_N}\right)^{\frac{1}{2}} \sinh\left((a_N b_N)^{\frac{1}{2}}\left(t_R^N - t^{N-1}\right)\right)$$

$$+ \zeta_{N3} \frac{b_{1N}}{b_N}\left(\cosh\left((a_N b_N)^{\frac{1}{2}}\left(t_R^N - t^{N-1}\right)\right) - 1\right)$$

$$- \gamma_{N3} \frac{a_{1N}}{(a_N b_N)^{\frac{1}{2}}} \sinh\left((a_N b_N)^{\frac{1}{2}}\left(t_R^N - t^{N-1}\right)\right)$$

$$- \gamma_{N3} a_{1N}\left(t - t_R^N\right) \quad .$$

So

$$\frac{\partial \hat{X}_1(t)}{\partial t_R^N} = X_1(t^{N-1}) \sinh\left((a_N b_N)^{\frac{1}{2}}\left(t_R^N - t^{N-1}\right)\right)(a_N b_N)^{\frac{1}{2}}$$

$$- Y_1(t^{N-1})\left(\frac{a_N}{b_N}\right)^{\frac{1}{2}} \cosh\left((a_N b_N)^{\frac{1}{2}}\left(t_R^N - t^{N-1}\right)\right)(a_N b_N)^{\frac{1}{2}}$$

$$+ \zeta_{N3} \frac{b_{1N}}{b_N} \sinh\left((a_N b_N)^{\frac{1}{2}}\left(t_R^N - t^{N-1}\right)\right)(a_N b_N)^{\frac{1}{2}}$$

$$- \gamma_{N3} \frac{a_{1N}}{(a_N b_N)^{\frac{1}{2}}} \cosh\left((a_N b_N)^{\frac{1}{2}}\left(t_R^N - t^{N-1}\right)\right)(a_N b_N)^{\frac{1}{2}}$$

$$+ \gamma_{N3} a_{1N}$$

$$= -\left(a_N b_N\right)^{\frac{1}{2}}\left[Y_1\left(t^{N-1}\right)\left(\frac{a_N}{b_N}\right)^{\frac{1}{2}} \cosh\left(\left(a_N b_N\right)^{\frac{1}{2}}\left(t_R^N - t^{N-1}\right)\right)\right.$$

$$- X_1\left(t^{N-1}\right) \sinh\left(\left(a_N b_N\right)^{\frac{1}{2}}\left(t_R^N - t^{N-1}\right)\right)$$

$$+ \gamma_{N3} \frac{a_{1N}}{\left(a_N b_N\right)^{\frac{1}{2}}}\left(\cosh\left(\left(a_N b_N\right)^{\frac{1}{2}}\left(t_R^N - t^{N-1}\right)\right) - 1\right)$$

$$\left. - \zeta_{N3} \frac{b_{1N}}{b_N} \sinh\left(\left(a_N b_N\right)^{\frac{1}{2}}\left(t_R^N - t^{N-1}\right)\right)\right]$$

$$= - a_N\left[Y_1\left(t^{N-1}\right) \cosh\left(\left(a_N b_N\right)^{\frac{1}{2}}\left(t_R^N - t^{N-1}\right)\right)\right.$$

$$- X_1\left(t^{N-1}\right)\left(\frac{b_N}{a_N}\right)^{\frac{1}{2}} \sinh\left(\left(a_N b_N\right)^{\frac{1}{2}}\left(t_R^N - t^{N-1}\right)\right)$$

$$+ \gamma_{N3} \frac{a_{1N}}{a_N}\left(\cosh\left(\left(a_N b_N\right)^{\frac{1}{2}}\left(t_R^N - t^{N-1}\right)\right) - 1\right)$$

$$\left. - \zeta_{N3} \frac{b_{1N}}{\left(a_N b_N\right)^{\frac{1}{2}}} \sinh\left(\left(a_N b_N\right)^{\frac{1}{2}}\left(t_R^N - t^{N-1}\right)\right)\right]$$

$$= - a_N Y_1\left(t_R^N\right) = 0 \quad .$$

We conclude $\hat{X}_1(t)$ and τ_B^N $\left(t_R^N < t_B^N \text{ and } t_R^N < t \leq \tau_B^N\right)$ are nondecreasing in ζ_{N3} and nonincreasing in γ_{N3}. In a similar fashion $\hat{Y}_1(t)$ and τ_R^N $\left(t_B^N < t_R^N \text{ and } t_B^N < t \leq \tau_R^N\right)$ are nondecreasing in γ_{N3} and nonincreasing in ζ_{N3}.

Suppose $\gamma_{N3} = 1$, then the payoff at stage N is as above. We have seen if

$$t^{N-1} \le t^N \le t_B^N \wedge t_R^N \quad ,$$

then $\hat{X}_1(t^N)$ and $-\hat{Y}_1(t^N)$ are nondecreasing in ζ_{N3}; thus $\hat{X}_1(t^N) - \hat{Y}_1(t^N)$ is maximized at $\zeta_{N3} = 1$. If $t_R^N < t_B^N$ and $t_R^N < t^N \le \tau_B^N$, then $\hat{X}_1(t^N)$ and τ_B^N are nondecreasing in ζ_{N3}, $\hat{Y}_1(t^N) = 0$, and t_R^N is nonincreasing in ζ_{N3}; thus $\hat{X}_1(t^N) - \hat{Y}_1(t^N)$ is maximized at $\zeta_{N3} = 1$. If $t_B^N < t_R^N$ and $t_B^N < t^N \le \tau_R^N$, then $\hat{X}_1(t^N) \ge 0$, t_B^N, and $-\hat{Y}_1(t^N)$ are nondecreasing in ζ_{N3}, while τ_R^N is nonincreasing in ζ_{N3}; thus $\hat{X}_1(t^N) - \hat{Y}_1(t^N)$ is maximized at $\zeta_{N3} = 1$.

In an analogous fashion we can show if $\zeta_{N3} = 1$, then $\hat{X}_1(t^N) - \hat{Y}_1(t^N)$ is minimized at $\gamma_{N3} = 1$. Thus we have proved

Theorem: At stage N an optimal strategy for B assigns all aircraft to BR ($\zeta_{N3} = 1$), and an optimal strategy for R assigns all aircraft to BR ($\gamma_{N3} = 1$). The value of the game, $V_N(\zeta_{N-1}, \gamma_{N-1})$, is

$$= X_1(t^{N-1})\left[\cosh\left((a_N b_N)^{\frac{1}{2}}(t^N - t^{N-1})\right) + \left(\frac{b_N}{a_N}\right)^{\frac{1}{2}} \sinh\left((a_N b_N)^{\frac{1}{2}}(t^N - t^{N-1})\right)\right]$$

$$- Y_1(t^{N-1})\left[\left(\frac{a_N}{b_N}\right)^{\frac{1}{2}} \sinh\left((a_N b_N)^{\frac{1}{2}}(t^N - t^{N-1})\right) + \cosh\left((a_N b_N)^{\frac{1}{2}}(t^N - t^{N-1})\right)\right]$$

$$+ b_{1N}\left[\frac{\cosh\left((a_N b_N)^{\frac{1}{2}}(t^N - t^{N-1})\right) - 1}{b_N} + \frac{\sinh\left((a_N b_N)^{\frac{1}{2}}(t^N - t^{N-1})\right)}{(a_N b_N)^{\frac{1}{2}}}\right]$$

$$- a_{1N}\left[\frac{\cosh\left((a_N b_N)^{\frac{1}{2}}(t^N - t^{N-1})\right) - 1}{a_N} + \frac{\sinh\left((a_N b_N)^{\frac{1}{2}}(t^N - t^{N-1})\right)}{(a_N b_N)^{\frac{1}{2}}}\right] \quad ,$$

$$\text{if} \quad t^N \le t_B^N \wedge t_R^N$$

$$= X_1(t^{N-1}) \cosh\left((a_N b_N)^{\frac{1}{2}}\left(t_R^N - t^{N-1}\right)\right) - Y_1(t^{N-1})\left(\frac{a_N}{b_N}\right)^{\frac{1}{2}} \sinh\left((a_N b_N)^{\frac{1}{2}}\left(t_R^N - t^{N-1}\right)\right)$$

$$+ \frac{b_{1N}}{b_N}\left[\cosh\left((a_N b_N)^{\frac{1}{2}}\left(t_R^N - t^{N-1}\right)\right) - 1\right]$$

$$- a_{1N}\left[\frac{\sinh\left((a_N b_N)^{\frac{1}{2}}\left(t_R^N - t^{N-1}\right)\right)}{(a_N b_N)^{\frac{1}{2}}} + t^N - t_R^N\right] ,$$

$$\text{if} \quad t_R^N < t_B^N \quad \text{and} \quad t_R^N < t^N \le \tau_B^N$$

$$= - Y_1(t^{N-1}) \cosh\left((a_N b_N)^{\frac{1}{2}}\left(t_B^N - t^{N-1}\right)\right) + X_1(t^{N-1})\left(\frac{b_N}{a_N}\right)^{\frac{1}{2}} \sinh\left((a_N b_N)^{\frac{1}{2}}\left(t_B^N - t^{N-1}\right)\right)$$

$$- \frac{a_{1N}}{a_N}\left[\cosh\left((a_N b_N)^{\frac{1}{2}}\left(t_B^N - t^{N-1}\right)\right) - 1\right]$$

$$+ b_{1N}\left[\frac{\sinh\left((a_N b_N)^{\frac{1}{2}}\left(t_B^N - t^{N-1}\right)\right)}{(a_N b_N)^{\frac{1}{2}}} + t^N - t_B^N\right] ,$$

$$\text{if} \quad t_B^N < t_R^N \quad \text{and} \quad t_B^N < t^N \le \tau_R^N$$

$$= 0, \quad \text{otherwise.}$$

A major difficulty is now evident. Recall that a_N, b_N, a_{1N}, and b_{1N}, as well as $X_1(t^{N-1})$ and $Y_1(t^{N-1})$, are functions of ζ_{N-1} and γ_{N-1}. Thus V_N is a complicated function of the strategies in stage N-1. In

fact, as a result of this functional dependence, a prodigious amount of analytical effort would be required simply to establish the behavior of V_N as a function of ζ_{N-1} or γ_{N-1}. Rather than expend this effort, it was judged more profitable to construct a more tractable model. In this model, attrition is given by difference equations and the payoff is a simple linear function.

IV MODEL III
(THE SURROGATE MODEL)

As a matter of convenience, we number the stages from the end of the game; i.e., stage 1 is the last stage, stage 2 is the second from the last stage, etc. Let $X_n(Y_n)$ be the number of B(R) aircraft available for assignment at the beginning of stage n. Let $X_{nq}(Y_{nq})$ be the number of B(R) aircraft assigned to region q (q = B,R,BR) at the beginning of stage n. An air allocation for B(R) in stage n is a vector $\underline{X}_n(\underline{Y}_n) \varepsilon \mathbb{R}^3$, where $\underline{X}_n = (X_{nB}, X_{nR}, X_{nBR})$ $(\underline{Y}_n = (Y_{nB}, Y_{nR}, Y_{nBR}))$ $X_{nB} + X_{nR} + X_{nBR} = X_n(Y_{nB} + Y_{nR} + Y_{nBR} = Y_n)$ and $X_{nq}(Y_{nq}) \geq 0$.

Let $\alpha_q^n\left(\beta_q^n\right)$ be the fraction of $X_{nq}(Y_{nq})$ suballocated to the airfield in region q, q = B,R. Let $x_n^i\left(y_n^i\right)$, $1 \leq i \leq 3$, be the order of battle of B(R)'s AN, CAS, and I mission, respectively, at the end of stage n. Let $x_n^i\left(y_n^i\right)$, i = 4,5 be the order of battle of B(R)'s AD vs AN, and AD vs I mission, respectively, at the end of stage n. Let $\hat{x}_n(\hat{y}_n)$ be the number of B(R) aircraft destroyed on the ground by R(B)'s AN mission survivors. Thus x_n^i, $\hat{x}_n$, y_n^i, $\hat{y}_n$ are functions of $\underline{X}_n$, $\underline{Y}_n$, α_q^n and β_q^n.

A. The Air and Ground War

1. Ground War and Logistic Pipelines

In an effort to circumvent the complexities in Model II, the explicit representation of the ground war was sacrificed and replaced by a simple linear function of the order of battle of B and R CAS missions:

$$aX_{nBR} - Y_{nBR}$$

$a > 0$ and $1 \leq n \leq N$.

2. Difference Equations for the Air War

The following system of difference equations is assumed to represent air attrition from stage n to n-1.

3. Airfield Neutralization

$$x_n^1 = 0 \vee \left(\alpha_R^n X_{nR} - \beta_R^n Y_{nR}\right) ,$$

$$y_n^4 = 0 \vee \left(\beta_R^n Y_{nR} - \alpha_R^n X_{nR}\right) ,$$

$$y_n^1 = 0 \vee \left(\beta_B^n Y_{nB} - \alpha_B^n X_{nB}\right) ,$$

$$x_n^4 = 0 \vee \left(\alpha_B^n X_{nB} - \beta_B^n Y_{nB}\right) ,$$

$$\hat{x}_n = X \wedge y_n^1$$

$$\hat{y}_n = Y \wedge x_n^1 .$$

4. Close Air Support

$$x_n^2 = 0 \vee \left(X_{nBR} - Y_{nBR}\right) ,$$

$$y_n^2 = 0 \vee \left(Y_{nBR} - X_{nBR}\right) .$$

5. Interdiction

$$x_n^3 = 0 \vee \left(\left(1-\alpha_R^n\right)X_{nR} - \left(1-\beta_R^n\right)Y_{nR}\right) ,$$

$$y_n^5 = 0 \vee \left(\left(1-\beta_R^n\right)Y_{nR} - \left(1-\alpha_R^n\right)X_{nR}\right) ,$$

$$y_n^3 = 0 \vee \left(\left(1-\beta_B^n\right)Y_{nB} - \left(1-\alpha_B^n\right)X_{nB}\right) ,$$

$$x_n^5 = 0 \vee \left(\left(1-\alpha_B^n\right)X_{nB} - \left(1-\beta_B^n\right)Y_{nB}\right) .$$

So $$X_{n-1} = 0 \vee \left[\sum_{1\le i\le 5} x_n^i - \hat{x}_n\right]$$

and $$Y_{n-1} = 0 \vee \left[\sum_{1\le i\le 5} y_n^i - \hat{y}_n\right] \quad .$$

B. <u>The Formulated Game</u>

B(R) picks $\underline{X}_n = (X_{nB}, X_{nR}, X_{nBR})$ $(\underline{Y}_n = (Y_{nB}, Y_{nR}, Y_{nBR}))$ to max(min) the payoff function

$$\sum_{1\le n\le N} \left[aX_{nBR} - Y_{nBR}\right]$$

so that for $1 \le n \le N$

$$X_{n-1} = 0 \vee \left[\sum_{1\le i\le 5} x_n^i - \hat{x}_n\right]$$

$$Y_{n-1} = 0 \vee \left[\sum_{1\le i\le 5} y_n^i - \hat{y}_n\right]$$

$$X_{nB} + X_{nR} + X_{nBR} = X_n$$

$$Y_{nB} + Y_{nR} + Y_{nBR} = Y_n$$

$$X_{nq} , Y_{nq} \ge 0 \quad .$$

C. <u>The Solution of the Game (A Beginning)</u>

Consider stage 1. Then for a choice of strategies for B and R the payoff is $M_1(\underline{X}_1, \underline{Y}_1) = aX_{1BR} - Y_{1BR}$. Obviously M_1 is maximized(minimized) for B(R) when $X_{1BR} = X_1(Y_{1BR} = Y_1)$. The value of the game is $V_1(X_1, Y_1) = aX_1 - Y_1$. Thus, we have shown

Theorem: At stage 1 an optimal strategy for B assigns all aircraft to BR $(X_{1BR} = X_1)$; an optimal strategy for R assigns all aircraft to BR $(Y_{1BR} = Y_1)$. The value of the game is $V_1(X_1, Y_1) = aX_1 - Y_1$.

Consider stage 2. Then for a choice of strategies for B and R, the payoff is

$$M_2(\underline{X}_2, \underline{Y}_2) = a(X_{2BR} + X_1) - (Y_{2BR} + Y_1)$$

$$= a\left(X_{2BR} + 0 \vee \left[\sum_{1\le i\le 5} x_2^i - \hat{x}_2\right]\right)$$

$$- \left(Y_{2BR} + 0 \vee \left[\sum_{1\le i\le 5} y_2^i - \hat{y}_2\right]\right) \quad .$$

For the remainder of this section we will usually suppress the subscript denoting the stage. We shall assume B is that player in stage 2 with larger resources, i.e., $X \ge Y$. (From symmetry and the solution for the case when $X \ge Y$, we may easily obtain the solution for $X \le Y$.) Thus $\hat{x} = y^1$ whence

$$x^4 - \hat{x} = \alpha_B X_B - \beta_B Y_B \quad .$$

Furthermore, it is not difficult to show that

$$y^4 - \hat{y} = \beta_R Y_R - \alpha_R X_R \wedge (Y + \beta_R Y_R) \quad .$$

In what follows we shall determine the conditions to be satisfied by a so that an optimal strategy for B assigns all aircraft to BR ($X_{BR} = X$), and an optimal strategy for R assigns all aircraft to BR ($Y_{BR} = Y$).

If $\underline{X}^o = (0,0,X)$ and $\underline{Y}^o = (0,0,Y)$, then $M_2\left(\underline{X}^o, \underline{Y}^o\right) = 2aX - (a+1)Y$. Let $\underline{Y}$ be any R strategy, then

$$M_2\left(\underline{X}^o, \underline{Y}\right) = a\left(X + 0 \vee [X - Y_{BR} - \beta_B Y_B]\right) - Y$$

$$= 2aX - Y - aY_{BR} - a\beta_B Y_B \quad .$$

Since $0 \le \beta_B \le 1$, it is obvious that $M_2\left(\underline{X}^o, \underline{Y}\right)$ is minimized at $Y_{BR} = Y$.

Whence $$\min_{\underline{Y}} M_2\left(\underline{X}^o, \underline{Y}\right) = 2aX - (1+a)Y$$

and this result is independent of $a > 0$.

Let $\underline{X}$ be any B strategy; then

$$M_2\left(\underline{X}, \underline{Y}^o\right) = a\left(X + 0 \vee (X_{BR}-Y)\right)$$
$$- \left(Y + 0 \vee \left[0 \vee (Y-X_{BR}) - \alpha_R X_R \wedge Y\right]\right) \quad .$$

There are several cases.

Case 1: $\alpha_R X_R \leq Y$, $X_{BR} \geq Y$.

Then $M_2\left(\underline{X}, \underline{Y}^o\right) = aX - (1+a)Y + aX_{BR}$.

Certainly $\max_{\underline{X}} M_2\left(\underline{X}, \underline{Y}^o\right) = 2aX - (1+a)Y$,

in this case, and it occurs when $X_{BR} = X$.

Thus $\underline{X}^o$ is optimal here.

Case 2: $\alpha_R X_R \leq Y$, $X_{BR} \leq Y$, and $X_{BR} + \alpha_R Y_R \geq Y$.

Then $M_2\left(\underline{X}, \underline{Y}^o\right) = aX - Y$.

Thus $\max_{\underline{X}} M_2\left(\underline{X}, \underline{Y}^o\right) = aX - Y$ in this case and any $\underline{X}$ in Case 2 is optimal here.

Case 3: $\alpha_R X_R \leq Y$, $X_{BR} \leq Y$ and $X_{BR} + \alpha_R X_R \leq Y$.

Then $M_2\left(\underline{X}, \underline{Y}^o\right) = aX - 2Y + X_{BR} + \alpha_R X_R$.

So $\max_{\underline{X}} M_2\left(\underline{X}, \underline{Y}^o\right) = aX - 3Y$ and a maximum occurs at $X_B = X - Y$, $X_{BR} = Y$.

Case 4: $\alpha_R X_R \geq Y$.

Then $M_2\left(\underline{X}, \underline{Y}^o\right) = aX - Y + a\left(0 \vee (X_{BR} - Y)\right)$.

Consequently, $\max\limits_{\underline{X}} M_2\left(\underline{X}, \underline{Y}^o\right) = aX - Y + a\left(0 \vee \left(X - \frac{1 + \alpha_R}{\alpha_R} Y\right)\right)$

and a maximum occurs at $X_R = Y/\alpha_R$, $X_{BR} = X - Y/\alpha_R$.

Note that there always exists an $\underline{X}$ in Cases 1, 2, and 3; while there exists an $\underline{X}$ in Case 4 if and only if $X \geq Y/\alpha_R$. Thus

$$\max_{\underline{X}} M_2\left(\underline{X}, \underline{Y}^o\right) = \begin{cases} \max\left[2aX - (1+a)Y,\ aX - Y,\ aX - 3Y\right] , \\ \qquad \text{if} \quad X < Y/\alpha_R , \\ \max \Big[2aX - (1+a)Y,\ aX - Y,\ aX - 3Y, \\ \qquad aX - Y + a\left(0 \vee \left(X - \frac{1 + \alpha_R}{\alpha_R} Y\right)\right)\Big] , \\ \qquad \text{if} \quad X \geq Y/\alpha_R . \end{cases}$$

Since $X \geq Y$, it follows that $2aX - (1+a)Y \geq aX - Y \geq aX - 3Y$. Furthermore,

$$2aX - (1+a)Y \geq aX - Y + a\left(0 \vee \left(X - \frac{1 + \alpha_R}{\alpha_R} Y\right)\right)$$

if and only if

$$X - Y \geq 0 \vee \left(X - \frac{1 + \alpha_R}{\alpha_R} Y\right) .$$

The inequality is obviously true when

$$X \leq \frac{1 + \alpha_R}{\alpha_R} Y .$$

When

$$X > \frac{1 + \alpha_R}{\alpha_R} Y ,$$

we conclude $$X - Y \geq X - \frac{1 + \alpha_R}{\alpha_R} Y$$

if and only if $$1 \leq \frac{1 + \alpha_R}{\alpha_R} \quad .$$

Whence $\max\limits_{\underline{X}} M_2\left(\underline{X}, \underline{Y}^o\right) = 2aX - (1+a)Y$ and the maximum occurs when $X_{BR} = X$.

So we have

$$M_2\left(\underline{X}^o, \underline{Y}^o\right) = 2aX - (1+a)Y \quad ,$$

and $$\min_{\underline{Y}} M_2\left(\underline{X}^o, \underline{Y}\right) = M_2\left(\underline{X}^o, \underline{Y}^o\right) = \max_{\underline{X}} M_2\left(\underline{X}, \underline{Y}^o\right) \quad .$$

Whence we conclude that a saddlepoint exists with $\underline{X}^o$ and $\underline{Y}^o$ optimal pure strategies.

Thus we have proven

Theorem: Suppose $X_2 \geq Y_2$, then at stage 2 an optimal strategy for B assigns all aircraft to BR ($X_{2BR} = X_2$), and an optimal strategy for R assigns all aircraft to BR ($Y_{2BR} = Y_2$). The value of the game is

$$V_2(X_2, Y_2) = 2aX_2 - (1+a)Y_2 \quad .$$

From symmetry we also have

Theorem: Suppose $X_2 \leq Y_2$, then at stage 2 an optimal strategy for B assigns all aircraft to BR ($X_{2BR} = X_2$), and an optimal strategy for R assigns all aircraft to BR ($Y_{2BR} = Y_2$). The value of the game is

$$V_2(X_2, Y_2) = (1+a)X_2 - 2Y_2 \quad .$$

Consider stage 3. For a choice of strategies for B and R the payoff is

$$M_3(\underline{X}_3, \underline{Y}_3) = aX_{3BR} - Y_{3BR} + V_2(X_2, Y_2)$$

$$= a\left(X_{3BR} + X_2 + 0 \vee \{X_2 - Y_2\}\right) - \left(Y_{3BR} + Y_2 + 0 \vee \{Y_2 - X_2\}\right)$$

$$= a\Bigg(X_{3BR} + 0 \vee \left[\sum_{1\le i\le 5} x_3^i - \hat{x}_3\right]$$

$$+ 0 \vee \left\{0 \vee \left[\sum_{1\le i\le 5} x_3^i - \hat{x}_3\right] - 0 \vee \left[\sum_{1\le i\le 5} y_3^i - \hat{y}_3\right]\right\}\Bigg)$$

$$- \Bigg(Y_{3BR} + 0 \vee \left[\sum_{1\le i\le 5} y_3^i - \hat{y}_3\right]$$

$$+ 0 \vee \left\{0 \vee \left[\sum_{1\le i\le 5} y_3^i - \hat{y}_3\right] - 0 \vee \left[\sum_{1\le i\le 5} x_3^i - \hat{x}_3\right]\right\}\Bigg) \quad .$$

We assume B is that player in stage 3 with larger resources, i.e., $X_3 \ge Y_3$. The case for $X_3 \le Y_3$ follows easily from symmetry.

In what follows we shall determine the conditions to be satisfied by a so that an optimal strategy for B assigns all aircraft to BR ($X_{3BR} = X_3$), and an optimal strategy for R assigns all aircraft to BR ($Y_{3BR} = Y_3$). For the remainder of this section we usually suppress the subscript denoting the stage.

If $\underline{X}^o = (0,0,X)$ and $\underline{Y}^o = (0,0,Y)$, then $M_3(\underline{X}^o, \underline{Y}^o) = 3aX - (2a+1)Y$. Let $\underline{Y}$ be any R strategy, then

$$M_3(\underline{X}^o, \underline{Y}) = 2aX - Y - a(Y_{BR} + \beta_B Y_B) + a(X - Y - \beta_B Y_B) \vee 0$$

$$- (Y + \beta_B Y_B - X) \vee 0 \quad .$$

There are two cases.

Case 1: $X \ge Y + \beta_B Y_B$.

Then $M_3(\underline{X}^o, \underline{Y}) = 3aX - (a+1)Y - aY_{BR} - 2\beta_B aY_B$.

If $2\beta_B \le 1$, $\min_{\underline{Y}} M_3(\underline{X}^o, \underline{Y}) = 3aX - (2a+1)Y$ and the minimum occurs when $Y_{BR} = Y$. If $2\beta_B > 1$,

$$\min_{\underline{Y}} M_3(\underline{X}^o, \underline{Y}) = 3aX - (a+1)Y - a\left(Y - Y \wedge \left(\frac{X-Y}{\beta_B}\right)\right) - 2\beta_B\, a\left(Y \wedge \left(\frac{X-Y}{\beta_B}\right)\right)$$

$$= 3aX - (2a+1)Y - a(2\beta_B - 1)\left(Y \wedge \left(\frac{X-Y}{\beta_B}\right)\right)$$

and the minimum occurs when

$$Y_B = Y \wedge \left(\frac{X-Y}{\beta_B}\right) \quad , \quad Y_{BR} = Y - Y \wedge \left(\frac{X-Y}{\beta_B}\right) \quad .$$

Case 2: $X \le Y + \beta_B Y_B$.

Then $M_3(\underline{X}^o, \underline{Y}) = (2a+1)X - 2Y - aY_{BR} - \beta_B(a+1)Y_B$.

If $a \ge \beta_B/(1-\beta_B)$, then

$$\min_{\underline{Y}} M_3(\underline{X}^o, \underline{Y}) = (2a+1)X - 2Y - a\left(Y - \frac{X-Y}{\beta_B}\right) - \beta_B(a+1)\left(\frac{X-Y}{\beta_B}\right)$$

$$= a(1 + 1/\beta_B)X - (a/\beta_B + 1)Y \quad ,$$

and the minimum occurs when

$$Y_B = \frac{X-Y}{\beta_B} \quad \text{and} \quad Y_{BR} = Y - \frac{X-Y}{\beta_B} \, .$$

If $a \le \beta_B/(1-\beta_B)$, then

$$\min_{\underline{Y}} M_3(\underline{X}^o, \underline{Y}) = (2a+1)X - 2Y - \beta_B(a+1)Y = (2a+1)X - (\beta_B a + \beta_B + 2)Y \quad ,$$

and the minimum occurs when $Y_B = Y$.

Note that there always exists $\underline{Y}$ in Case 1, namely $\underline{Y} = (0, Y_R, Y_{BR})$, where $Y_R + Y_{BR} = Y$, while there exists $\underline{Y}$ in Case 2 if and only if $X \le (1+\beta_B)Y$. We may conclude therefore that

$$\min_{\underline{Y}} M_3(\underline{X}^o, \underline{Y}) = 3aX - (2a+1)Y \quad , \quad \text{if}$$

$$X \geq (1+\beta_B)Y \quad \text{and} \quad 2\beta_B \leq 1 \quad ,$$

$$= 3aX - ((1+2\beta_B)a + 1)Y \quad , \quad \text{if}$$

$$X \geq (1+\beta_B)Y \quad \text{and} \quad 2\beta_B > 1 \quad ,$$

$$= \min \left[3aX - (2a+1)Y, \ a(1+1/\beta_B)X - (a/\beta_B+1)Y\right] \quad , \quad \text{if}$$

$$X \leq (1+\beta_B)Y, \ 2\beta_B \leq 1, \quad \text{and} \quad a \geq \beta_B/(1-\beta_B) \quad ,$$

$$= \min \left[3aX - (2a+1)Y, \ (2a+1)X - (\beta_B a + \beta_B + 2)Y\right], \quad \text{if}$$

$$X \leq (1+\beta_B)Y, \ 2\beta_B \leq 1, \quad \text{and} \quad a < \beta_B/(1-\beta_B) \quad ,$$

$$= a(1+1/\beta_B)X - (a/\beta_B+1)Y \quad , \quad \text{if}$$

$$X \leq (1+\beta_B)Y, \ 2\beta_B > 1, \quad \text{and} \quad a \geq \beta_B/(1-\beta_B) \quad ,$$

$$= \min \left[a(1+1/\beta_B)X - (a/\beta_B+1)Y \quad , \right.$$

$$\left. (2a+1)X - (\beta_B a + \beta_B + 2)Y\right] \quad , \quad \text{if}$$

$$X \leq (1+\beta_B)Y, \ 2\beta_B > 1, \quad \text{and} \quad a < \beta_B/(1-\beta_B) \quad .$$

It is easy to see that $3aX - (2a+1)Y \leq a(1+1/\beta_B)X - (a/\beta_B+1)Y$ if and only if $2\beta_B \leq 1$.

Suppose that $X \le (1+\beta_B)Y$, $2\beta_B \le 1$, and $a < \beta_B/(1-\beta_B)$. Then $a < 1$ and $3aX - (2a+1)Y \le (2a+1)X - (\beta_B a + \beta_B + 2)Y$ if and only if $((\beta_B - a(1-\beta_B))/(1-a) + 1)Y \le X$. Yet, $(\beta_B - a(1-\beta_B))/(1-a) \le \beta_B$ if and only if $\beta_B \le 1 - \beta_B$, which is true by assumption. Whence

$$\min_{\underline{Y}} M_3\left(\underline{X}^o, \underline{Y}\right) = 3aX - (2a+1)Y$$

if any one of the following three conditions is satisfied:

(1) $(1+\beta_B)Y \le X$, and $2\beta_B \le 1$;

(2) $Y \le X < (1+\beta_B)Y$, $2\beta_B \le 1$, and $a \ge \beta_B/(1-\beta_B)$;

(3) $((\beta_B - a(1-\beta_B))/(1-a) + 1)Y \le X < (1+\beta_B)Y$, $2\beta_B \le 1$, and $a < \beta_B/(1-\beta_B)$.

Let $\underline{X}$ be any B strategy, then

$$\begin{aligned} M_3\left(\underline{X}, \underline{Y}^o\right) = aX - Y + a\Big(&0 \vee (X_{BR}-Y) \\ &+ 0 \vee \{X_B + X_R + 0 \vee (X_{BR}-Y) \\ &- 0 \vee [0 \vee (Y-X_{BR}) - Y \wedge \alpha_R X_R]\}\Big) \\ &- \Big(0 \vee [0 \vee (Y-X_{BR}) - Y \wedge \alpha_R X_R] \\ &+ 0 \vee \{0 \vee [0 \vee (Y-X_{BR}) - Y \wedge \alpha_R X_R] \\ &- X_B - X_R - 0 \vee (X_{BR}-Y)\}\Big) \quad . \end{aligned}$$

There are several cases.

Case 1: $\alpha_R X_R \le Y$, $X_{BR} \ge Y$.

Then $M_3\left(\underline{X}, \underline{Y}^o\right) = 2aX - (2a+1)Y + aX_{BR}$,

$\max_{\underline{X}} M_3(\underline{X}, \underline{Y}^o) = 3aX - (2a+1)Y$, and the maximum occurs when $X_{BR} = X$.

Case 2: $\alpha_R X_R \le Y$, $X_{BR} \le Y$, and $X_{BR} + \alpha_R X_R \ge Y$.

Then $M_3(\underline{X}, \underline{Y}^o) = 2aX - Y - aX_{BR}$.

If $\alpha_R X \ge Y$, then $\max_{\underline{X}} M_3(\underline{X}, \underline{Y}^o) = 2aX - Y$ and the maximum occurs at $X_B = X - Y/\alpha_R$, $X_R = Y/\alpha_R$. If $\alpha_R X \le Y$, then $\max_{\underline{X}} M_3(\underline{X}, \underline{Y}^o) = ((2-\alpha_R)aX - (a+1-\alpha_R)Y)/(1-\alpha_R)$ and the maximum occurs at $X_R = \dfrac{X-Y}{1-\alpha_R}$, $X_{BR} = \dfrac{Y-\alpha_R X}{1-\alpha_R}$.

Case 3: $\alpha_R X_R \le Y$, $X_{BR} \le Y$, and $X_{BR} + \alpha_R X_R \le Y$.

Then $M_3(\underline{X}, \underline{Y}^o) = 2aX - (a+2)Y + X_{BR} + (a+1)\alpha_R X_R$.

Suppose $\alpha_R X \ge Y$. Then $\max_{\underline{X}} M_3(\underline{X}, \underline{Y}^o) = 2aX - Y$ and the maximum occurs at $X_B = X - Y/\alpha_R$, $X_R = Y/\alpha_R$. Suppose $\alpha_R X \le Y$. Then if $a \le (1-\alpha_R)/\alpha_R$, $\max_{\underline{X}} M_3(\underline{X}, \underline{Y}^o) = ((2-\alpha_R)aX - (a+1-\alpha_R)Y)/(1-\alpha_R)$, and the maximum occurs at $X_R = (X-Y)/(1-\alpha_R)$, $X_{BR} = (Y-\alpha_R X)/(1-\alpha_R)$.

If $a > (1-\alpha_R)/\alpha_R$, then $\max_{\underline{X}} M_3(\underline{X}, \underline{Y}^o) = (2a + \alpha_R a + \alpha_R)X - (a+2)Y$ and the maximum occurs at $X_R = X$.

Case 4: $\alpha_R X_R \ge Y$ and $X_{BR} \ge Y$.

Then $M_3(\underline{X}, \underline{Y}^o) = 2aX - (2a+1)Y + aX_{BR}$,

$\max_{\underline{X}} M_3(\underline{X}, \underline{Y}^o) = 3aX - (2a + a/\alpha_R + 1)Y$, and the maximum occurs at

$X_R = Y/\alpha_R$, $X_{BR} = X - Y/\alpha_R$.

Case 5: $\alpha_R X_R \geq Y$ and $X_{BR} \leq Y$.

Then $M_3(\underline{X}, \underline{Y}^o) = 2aX - Y - aX_{BR}$,

$\max_{\underline{X}} M_3(\underline{X}, \underline{Y}^o) = 2aX - Y$, and the maximum occurs at $X_R = X$.

Note that there always exists $\underline{X}$ in Cases 1, 2, and 3, although there exists $\underline{X}$ in Case 4 if and only if $X \geq (1+1/\alpha_R)Y$ and there exists $\underline{X}$ in Case 5 if and only if $X \geq Y/\alpha_R$.

We may conclude

$$\max_{\underline{X}} M_3(\underline{X}, \underline{Y}^o) = \max\left[3aX - (2a+1)Y,\ 2aX - Y,\ 3aX - (2a + a/\alpha_R + 1)Y\right] ,$$

$$\text{if} \quad (1+1/\alpha_R)Y \leq X ,$$

$$= \max\left[3aX - (2a+1)Y,\ 2aX - Y\right] ,$$

$$\text{if} \quad Y/\alpha_R \leq X < (1+1/\alpha_R)Y ,$$

$$= \max\left[3aX - (2a+1)Y,\ ((2-\alpha_R)aX - (a+1-\alpha_R)Y)/(1-\alpha_R)\right] ,$$

$$\text{if} \quad X < Y/\alpha_R \quad \text{and} \quad a \leq (1-\alpha_R)/\alpha_R ,$$

$$= \max\left[3aX - (2a+1)Y,\ ((2-\alpha_R)aX - (a+1-\alpha_R)Y)/(1-\alpha_R) ,\right.$$

$$\left.(2a + \alpha_R a + \alpha_R)X - (a+2)Y\right] ,$$

$$\text{if } X < Y/\alpha_R \quad \text{and} \quad a > (1-\alpha_R)/\alpha_R .$$

One may easily show:

$$3aX - (2a+1)Y \geq 2aX - Y \quad \text{if and only if} \quad X \geq 2Y \quad ;$$

$$3aX - (2a+1)Y \geq 3aX - (2a + a/\alpha_R + 1)Y \quad \text{if and only if}$$

$$aY/\alpha_R \geq 0 \quad ;$$

$$3aX - (2a+1)Y \geq ((2-\alpha_R)aX - (a+1-\alpha_R)Y)/(1-\alpha_R) \quad \text{if and only if}$$

$$(1-2\alpha_R)X \geq (1-2\alpha_R)Y \quad \text{if and only if} \quad 1 \geq 2\alpha_R \quad .$$

Now suppose $a > (1-\alpha_R)/\alpha_R$ and $\alpha_R \leq 1/2$. It is obvious that $3aX - (2a+1)Y \geq (2a+\alpha_R a+\alpha_R)X - (a+2)Y$ if and only if $(a(1-\alpha_R) - \alpha_R)X \geq (a-1)Y$. Then $a > (1-\alpha_R)/\alpha_R$ if and only if $a - (a+1)\alpha_R < a - 1$ and if and only if $\alpha_R > 1/(a+1)$. Since $\alpha_R \leq 1/2$, we must have $1/(a+1) < 1/2$ from which we infer $a > 1$. Because $\alpha_R \leq 1/2$ implies that $\alpha_R/(1-\alpha_R) \leq 1$, we also infer $a > \alpha_R/(1-\alpha_R)$. Thus $(a(1-\alpha_R) - \alpha_R)X \geq (a-1)Y$ if and only if $X \geq (a-1)Y/(a(1-\alpha_R) - \alpha_R)$. Thus $3aX - (2a+1)Y \geq (2a+\alpha_R a+\alpha_R)X - (a+2)Y$, if $a > (1-\alpha_R)/\alpha_R$, $\alpha_R < 1/2$, and $X \geq (a-1)Y/(a(1-\alpha_R) - \alpha_R)$. Suppose $X < Y/\alpha_R$ is true. Then necessarily $(a-1)/(a(1-\alpha_R) - \alpha_R) < 1/\alpha_R$. This inequality is true if and only if $\alpha_R < 1 - \alpha_R$, and $\alpha_R < 1 - \alpha_R$ if and only if $\alpha_R < 1/2$.

Whence $\max_{\underline{X}} M_3\left(\underline{X}, \underline{Y}^o\right) = 3aX - (2a+1)Y$ if any one of the following three conditions is satisfied:

(4) $(2 \vee (1/\alpha_R))Y \leq X$;

(5) $Y \leq X < Y/\alpha_R$, $2\alpha_R \leq 1$, and $a \leq (1-\alpha_R)/\alpha_R$;

(6) $(a-1)Y/(a(1-\alpha_R) - \alpha_R) \leq X < Y/\alpha_R$, $2\alpha_R < 1$, and $a > (1-\alpha_R)/\alpha_R$.

We have derived conditions, involving X, Y, a, and β_B, that ensure, if satisfied, $\min_{\underline{Y}} M_3\left(\underline{X}^o, \underline{Y}\right) = M_3\left(\underline{X}^o, \underline{Y}^o\right)$. We have also derived conditions, involving X, Y, a, and α_R, that ensure, if satisfied, $\max_{\underline{X}} M_3\left(\underline{X}, \underline{Y}^o\right) = M_3\left(\underline{X}^o, \underline{Y}^o\right)$. The conditions that must be satisfied so that

$$\min_{\underline{Y}} M_3(\underline{X}^o, \underline{Y}) = M_3(\underline{X}^o, \underline{Y}^o) = \max_{\underline{X}} M_3(\underline{X}, \underline{Y}^o)$$

follow immediately from the conditions derived above. Thus

$$\min_{\underline{Y}} M_3(\underline{X}^o, \underline{Y}) = M_3(\underline{X}^o, \underline{Y}^o) = \max_{\underline{X}} M_3(\underline{X}, \underline{Y}^o)$$

if any one of the following five conditions is satisfied:

(7) $(2 \vee (1/\alpha_R))Y \le X, \quad 2\beta_B \le 1$;

(8) $(1+\beta_B)Y \le X < Y/\alpha_R, \; 2\beta_B \le 1, \; 2\alpha_R \le 1,$ and

$$0 < a \le (1-\alpha_R)/\alpha_R \quad ;$$

(9) $Y \le X < (1+\beta_B)Y, \; 2\beta_B \le 1, \; 2\alpha_R \le 1,$ and

$$\beta_B/(1-\beta_B) \le a \le (1-\alpha_R)/\alpha_R \quad ;$$

(10) $(a-1)Y/(a-\alpha_R(a+1)) \le X < (1+\beta_B)Y, \; 2\beta_B \le 1, \; 2\alpha_R < 1,$ and

$$(1-\alpha_R)/\alpha_R < a \quad ;$$

(11) $((\beta_B(a+1) - a)/(1-a) + 1)Y \le X < (1+\beta_B)Y, \; 2\beta_B \le 1, \; 2\alpha_R \le 1,$

and $0 < a < \beta_B/(1-\beta_B)$.

So, if any one of conditions 7 to 11 is satisfied, and $X \ge Y$, a saddle-point exists with $\underline{X}^o$ and $\underline{Y}^o$ optimal pure strategies.

Suppose $X \le Y$. In this case $M_3(\underline{X}^o, \underline{Y}^o) = (a+2)X - 3Y$. By symmetry we may conclude

$$\min_{\underline{Y}} M_3(\underline{X}^o, \underline{Y}) = M_3(\underline{X}^o, \underline{Y}^o) = \max_{\underline{X}} M_3(\underline{X}, \underline{Y}^o)$$

if any one of the following five conditions is satisfied:

(7′) $(2 \vee (1/\beta_B))X \le Y, \; 2\alpha_R \le 1$;

(8′) $(1+\alpha_R)X \le Y < X/\beta_B, \; 2\alpha_R \le 1, \; 2\beta_B \le 1,$ and

$$\beta_B/(1-\beta_B) \le a \quad ;$$

(9′) $X \le Y < (1+\alpha_R)X$, $2\alpha_R \le 1$, $2\beta_B \le 1$, and

$$\beta_B/(1-\beta_B) \le a \le (1-\alpha_R)/\alpha_R \quad ;$$

(10′) $(1-a)X/(1-\beta_B(1+a)) \le Y < (1+\alpha_R)X$, $2\alpha_R \le 1$, $2\beta_B < 1$, and

$$0 < a < \beta_B/(1-\beta_B) \quad ;$$

(11′) $((\alpha_R(a+1) - 1)/(a-1) + 1)X \le Y < (1+\alpha_R)X$, $2\alpha_R \le 1$, $2\beta_B \le 1$,

$$\text{and} \quad (1-\alpha_R)/\alpha_R < a \quad .$$

Whence, if any one of conditions 7′ to 11′ is satisfied, and $X \le Y$, a saddlepoint exists with $\underline{X}^o$ and $\underline{Y}^o$ optimal pure strategies. Thus we have proven:

Theorem: Suppose, at stage 3, $X_3 \ge Y_3$ and any one of conditions 7 to 11 is satisfied, then an optimal strategy for B assigns all aircraft to BR ($X_{3BR} = X_3$), an optimal strategy for R assigns all aircraft to BR ($Y_{3BR} = Y_3$), and the value of the game is $V_3(X_3, Y_3) = 3aX_3 - (2a+1)Y_3$.

Theorem: Suppose, at stage 3, $X_3 \le Y_3$ and any one of conditions 7′ to 11′ is satisfied, then an optimal strategy for B assigns all aircraft to BR ($X_{3BR} = X_3$), an optimal strategy for R assigns all aircraft to BR ($Y_{3BR} = Y_3$), and the value of the game is $V_3(X_3, Y_3) = (a+2)X_3 - 3Y_3$.

Pure optimal strategies may not exist in stage 3 if any one of the following 10 conditions is satisfied:

(12) $Y \le X$, and $1 < 2\beta_B$;

(13) $Y \le X < 2Y$, $2\beta_B \le 1$, and $1 < 2\alpha_R$;

(14) $Y \le X < ((\beta_B(a+1) - a)/(1-a) + 1)Y$, $2\beta_B \le 1$, $2\alpha_R \le 1$, and

$$0 < a < \beta_B/(1-\beta_B) \quad ;$$

(15) $Y \le X < (a-1)Y/(a-\alpha_R(a+1))$, $2\beta_B \le 1$, $2\alpha_R < 1$, and

$$(1-\alpha_R)/\alpha_R < a \quad ;$$

(16) $(1+\beta_B)Y \leq X < Y/\alpha_R$, $2\beta_B \leq 1$, $2\alpha_R < 1$, and $(1-\alpha_R)/\alpha_R < a$;

(12′) $X \leq Y$, and $1 < 2\alpha_R$;

(13′) $X \leq Y < 2X$, $1 < 2\beta_B$, and $2\alpha_R \leq 1$;

(14′) $X \leq Y < ((\alpha_R(a+1) - 1)/(a-1) + 1)X$, $2\beta_B \leq 1$, $2\alpha_R \leq 1$, and

$(1-\alpha_R)/\alpha_R < a$;

(15′) $X \leq Y < (1-a)X/(1-\beta_B(a+1))$, $2\beta_B < 1$, $2\alpha_R \leq 1$, and

$0 < a < \beta_B/(1-\beta_B)$;

(16′) $(1+\alpha_R)X \leq Y < X/\beta_B$, $2\beta_B < 1$, $2\alpha_R \leq 1$, and $0 < a < \beta_B/(1-\beta_B)$.

Because of analytic difficulties encountered in attempting to develop a closed form solution to the N-stage game given in IV-B, it was not possible to proceed with the research beyond this point due to the practical constraints of the present program. However, it is believed that this work will provide a practical basis for continuation of this research by analysts engaged in formulating solutions to this and similar problems. In this regard, it is believed that a firm foundation has been laid for any future research in this area.

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